BETHE EQUATION AT $q = 0$, MÖBIUS INVERSION FORMULA, AND WEIGHT MULTIPLICITIES: II. X_n CASE

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Abstract. We study a family of power series characterized by a system of recursion relations (Q-system) with a certain convergence property. We show that the coefficients of the series are expressed by the numbers which formally count the off-diagonal solutions of the $U_q(X_n^{(1)})$ Bethe equation at $q = 0$. The series are conjectured to be the X_n-characters of a certain family of irreducible finite-dimensional $U_q(X_n^{(1)})$-modules which we call the KR (Kirillov-Reshetikhin) modules. Under the above conjecture, these coefficients give a formula of the weight multiplicities of the tensor products of the KR modules, which is also interpreted as the formal completeness of the XXZ-type Bethe vectors.

1. Introduction

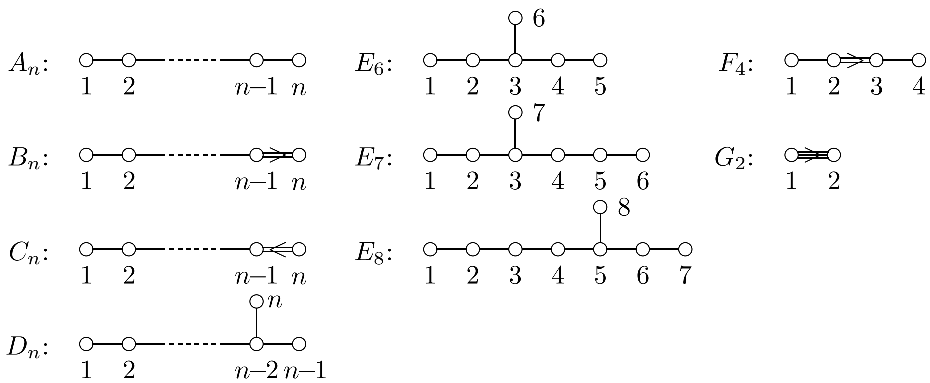
1.1. **Background.** Since the early days of the algebraic Bethe ansatz approach to the integrable lattice models, the finite-dimensional representations of the affine quantum group $U_q(X_n^{(1)})$ and its sister, Yangian $Y(X_n)$, have attracted much attention. The categories of finite-dimensional modules of $U_q(X_n^{(1)})$ and $Y(X_n)$ are equivalent [CP, D]. The case $X_n = A_n$ is rather well understood [CP, Ar]. Also, a general theory of the character has been constructed [FR, Kn]. However, no universal description of the character, like the Weyl character formula, is known so far.

In the series of works [K1, K2, K3, KR], Kirillov and Reshetikhin focused attention on a special class of $Y(X_n)$-modules. We call them (and the corresponding $U_q(X_n^{(1)})$-modules) the *KR modules*. The KR modules of $Y(X_n)$ (resp. $U_q(X_n^{(1)})$) provide natural generalizations of the well-known XXX (resp. XXZ) spin chain. They found that the formal counting of the Bethe vectors of the XXX-type spin chains, with the hypothesis of the completeness of the Bethe vectors, leads to a remarkable conjectural formula for the *multiplicity of the X_n-irreducible components* in the tensor products of the KR modules of $Y(X_n)$.

In this paper we show an analogous phenomenon occurs also for the XXZ-type case. In contrast with the XXX-type case, however, the formal counting of the Bethe vectors of the XXZ-type spin chains now leads to a conjectural formula for the *weight multiplicity* in the tensor products of the KR modules of $U_q(X_n^{(1)})$. We hope that these formulae will guide us to the proper understanding of the KR modules. This paper is the continuation of Part I [KN], where the case $X_n = A_1$ is described. Below we shall formulate and explain our problem more precisely.

1.2. **KR modules.** Let X_n be one of the finite-dimensional simple Lie algebras over $\mathbb{C}$ and $U_q(X_n^{(1)})$ be the non-twisted quantum affine algebra associated to X_n

Table 1. Dynkin diagrams



without the derivation operator. Let α_a and Λ_a $(a = 1, \dots, n)$ be the simple roots and fundamental weights of X_n. We enumerate the vertices of the Dynkin diagram as in Table 1. Let $(\cdot|\cdot)$ be the standard bilinear form normalized as $(\alpha|\alpha) = 2$ for a long root α. We set $t_a = 2/(\alpha_a|\alpha_a) \in \{1, 2, 3\}$. The Cartan matrix is $C_{ab} = t_a(\alpha_a|\alpha_b)$, and $\alpha_a = \sum_{b=1}^{n} C_{ba}\Lambda_b$.

The irreducible finite-dimensional $U_q(X_n^{(1)})$-modules are parameterized by n-tuples of polynomials (*Drinfeld polynomials*) $(P_b)_{b=1}^n$ with constant terms 1 [CP]. Here we follow the convention in [FR].

Definition 1.1. For each $a \in \{1, \dots, n\}$, $m \in \{1, 2, \dots\}$, and $u \in \mathbb{C}$, the irreducible finite-dimensional $U_q(X_n^{(1)})$-module whose Drinfeld polynomials (P_b) are

$$P_b(v) = \begin{cases} \prod_{j=1}^{m}(1 - vq^u q^{(m-2j+1)/t_a}) & b = a \\ 1 & \text{otherwise} \end{cases} \tag{1.1}$$

is called a *KR (Kirillov-Reshetikhin) module* and denoted by $W_m^{(a)}(u)$.

For $m = 1$, the KR modules are the fundamental modules. Through the injection $U_q(X_n) \to U_q(X_n^{(1)})$, any $U_q(X_n^{(1)})$-module W is regarded as a $U_q(X_n)$-module.

Lemma 1.2. *For any $u,\ u' \in \mathbb{C}$, $W_m^{(a)}(u)$ and $W_m^{(a)}(u')$ are equivalent as $U_q(X_n)$-modules.*

Proof. It is well known that $W_m^{(a)}(u')$ is obtained from $W_m^{(a)}(u)$ by a pull-back of an automorphism σ of $U_q(X_n^{(1)})$ which preserves $U_q(X_n)$. $\square$

In view of Lemma 1.2, let us write the common underlying $U_q(X_n)$-module for the family $W_m^{(a)}(u)$ as $W_m^{(a)}$. In general, there is a natural identification of a $U_q(X_n)$-module V with an X_n-module through the limit $q \to 1$. With this identification, we use the X_n-weight and X_n-character to describe V, instead of $U_q(X_n)$-weight and $U_q(X_n)$-character. For example, $\mathrm{ch}\, W_m^{(a)}$ means the X_n-character of $W_m^{(a)}$. The X_n-weight of the highest weight vector of $W_m^{(a)}(u)$ is $m\Lambda_a$.

1.3. **The series $Q_m^{(a)}$ and Kirillov-Reshetikhin's conjecture.** Let $x_a = e^{\Lambda_a}$ and $y_a = e^{-\alpha_a}$, $a = 1, \dots, n$, be the formal exponentials of the fundamental weights and (the minus of) the simple roots of X_n. With the multivariable notation

$x = (x_a)_{a=1}^n$ and $y = (y_a)_{a=1}^n$, we write the relation $y_a = \prod_{b=1}^n x_b^{-(\alpha_a|\alpha_b)t_b}$ as $y = y(x)$. Its inverse $x = x(y)$ involves fractional powers for some X_n. Let $\mathbb{C}[[y]]$ be the ring of (formal) power series of $y = (y_a)_{a=1}^n$ with the standard topology. For a power series $f(y)$, $f(y(x))$ is a Laurent series of x. Conversely, for a Laurent series $f(x)$, $f(x(y))$ is a Puiseax (fractional Laurent) series of y, in general.

Let H denote the index set

$$H = \{\, (a,m) \mid a \in \{1,\dots,n\},\ m \in \{1,2,\dots\}\,\}, \tag{1.2}$$

and for (a,m), $(b,k) \in H$, we define

$$B_{am,bk} = 2\min(t_b m, t_a k) - \min(t_b m, t_a(k+1)) - \min(t_b m, t_a(k-1)). \tag{1.3}$$

The following series are our main interest in this paper.

Theorem-Definition 1.3. *Let $(\tilde{Q}_m^{(a)}(y))_{(a,m)\in H}$ be the unique family of the invertible power series of y which satisfies ($\tilde{Q}$-I) and ($\tilde{Q}$-II):*

($\tilde{Q}$-I). (Q-system) Let $\tilde{Q}_0^{(a)}(y) = 1$. For $m = 1, 2, \dots$,

(1.4)
$$(\tilde{Q}_m^{(a)}(y))^2 = \tilde{Q}_{m+1}^{(a)}(y)\tilde{Q}_{m-1}^{(a)}(y) + y_a^m (\tilde{Q}_m^{(a)}(y))^2 \prod_{(b,k)\in H} (\tilde{Q}_k^{(b)}(y))^{-(\alpha_a|\alpha_b)B_{am,bk}}.$$

($\tilde{Q}$-II). (convergence property) The limit $\lim_{m\to\infty} \tilde{Q}_m^{(a)}(y)$ exists in $\mathbb{C}[[y]]$.
Let $Q_m^{(a)}(x) := x_a^m \tilde{Q}_m^{(a)}(y(x))$. Equivalently, $(Q_m^{(a)}(x))_{(m,a)\in H}$ is the unique family of the Laurent series of x which satisfies (Q-I) and (Q-II):

(Q-I). (Q-system) Let $Q_0^{(a)}(x) = 1$. For $m = 1, 2, \dots$,

$$(Q_m^{(a)}(x))^2 = Q_{m+1}^{(a)}(x)Q_{m-1}^{(a)}(x) + (Q_m^{(a)}(x))^2 \prod_{(b,k)\in H} (Q_k^{(b)}(x))^{-(\alpha_a|\alpha_b)B_{am,bk}}. \tag{1.5}$$

(Q-II). (convergence property) $\tilde{Q}_m^{(a)}(y) := x_a^{-m} Q_m^{(a)}(x)|_{x=x(y)}$ is an invertible power series of y, and the limit $\lim_{m\to\infty} \tilde{Q}_m^{(a)}(y)$ exists in $\mathbb{C}[[y]]$.

The existence and the uniqueness of $(\tilde{Q}_m^{(a)}(y))$, together with two explicit expressions, will be shown in Theorem 5.3, which is our key theorem.

Remark 1.4. The equivalence of the relations ($\tilde{\text{Q}}$-I) and (Q-I) is easily seen with (A.10). The product in the RHSs of ($\tilde{\text{Q}}$-I) and (Q-I) are actually finite products (Proposition A.1 (i)). From given invertible power series $\tilde{Q}_1^{(1)}, \dots, \tilde{Q}_1^{(n)}$, the relation ($\tilde{\text{Q}}$-I) recursively determines all the other invertible power series $\tilde{Q}_m^{(a)}$ (Proposition A.2).

Example 1.5. For A_1, the relation (1.4) becomes

$$(\tilde{Q}_m^{(1)}(y))^2 = \tilde{Q}_{m+1}^{(1)}(y)\tilde{Q}_{m-1}^{(1)}(y) + y_1^m. \tag{1.6}$$

It is easy to check that $\tilde{Q}_m^{(1)}(y) = \sum_{j=0}^m y_1^j$ satisfies the relation (1.6) and that $\lim_{m\to\infty} \tilde{Q}_m^{(1)}(y) = \sum_{j=0}^\infty y_1^j$. Therefore, $Q_m^{(1)}(x) = x_1^m + x_1^{m-2} + \cdots + x_1^{-m}$, which is the irreducible A_1-character with highest weight $m\Lambda_1$.

Let $\chi(\lambda)$ be the irreducible X_n-character with highest weight λ. The following two theorems were first proved for A_n by [K2], then generalized for the rest by [HKOTY]:

Theorem 1.6 ([K2, HKOTY]). *For X_n of classical type, i.e., $X_n = A_n$, B_n, C_n, and D_n, $Q^{(a)}_m$'s in (1.7)–(1.10) satisfy (Q-I) and (Q-II).*

$$A_n: \quad Q^{(a)}_m = \chi(m\Lambda_a), \tag{1.7}$$

$$B_n: \quad Q^{(a)}_m = \sum \chi(k_{a_0}\Lambda_{a_0} + k_{a_0+2}\Lambda_{a_0+2} + \cdots + k_a\Lambda_a), \tag{1.8}$$

$$C_n: \quad Q^{(a)}_m = \begin{cases} \sum \chi(k_1\Lambda_1 + k_2\Lambda_2 + \cdots + k_a\Lambda_a) & 1 \le a \le n-1 \\ \chi(m\Lambda_n) & a = n, \end{cases} \tag{1.9}$$

$$D_n: \quad Q^{(a)}_m = \begin{cases} \sum \chi(k_{a_0}\Lambda_{a_0} + k_{a_0+2}\Lambda_{a_0+2} + \cdots + k_a\Lambda_a) & 1 \le a \le n-2 \\ \chi(m\Lambda_a) & a = n-1, n. \end{cases} \tag{1.10}$$

In (1.8) and (1.10), $a_0 = 0$ or 1 with $a_0 \equiv a \bmod 2$, $\Lambda_0 = 0$, and the sum is taken over non-negative integers $k_{a_0}, \dots, k_a$ that satisfy $t_a(k_{a_0} + \cdots + k_{a-2}) + k_a = m$. In (1.9), the sum is taken over non-negative integers $k_1, \dots, k_a$ that satisfy $k_1 + \cdots + k_a \le m$, $k_b \equiv m\delta_{ab} \bmod 2$.

Because of the uniqueness, $Q^{(a)}_m(x)$ in Theorem 1.6 coincides with the one in Definition 1.3. In particular, $Q^{(a)}_m(x)$ is $\mathcal{W}$ invariant for any X_n of classical type, where $\mathcal{W}$ is the Weyl group of X_n.

Theorem 1.7 ([K2, HKOTY]). *Let X_n be arbitrary. If $Q^{(a)}_m(x)$ is $\mathcal{W}$ invariant for any $(a, m) \in H$, then*

$$Q^{(a)}_m = \sum_{\lambda \in P_+} \left\{ \sum_{N \in \mathcal{N}_\lambda} \prod_{(b,k) \in H} \binom{P^{(b)}_k + N^{(b)}_k}{N^{(b)}_k} \right\} \chi(\lambda), \tag{1.11}$$

where P_+ is the set of the dominant integral weights of X_n, and

$$P^{(b)}_k = \min(k, m)\delta_{ab} - \sum_{(c,j) \in H} (\alpha_b|\alpha_c) \min(t_c k, t_b j) N^{(c)}_j, \tag{1.12}$$

$$\mathcal{N}_\lambda = \{\, N = (N^{(b)}_k)_{(b,k) \in H} \mid N^{(b)}_k \in \mathbb{Z}_{\ge 0},\ m\Lambda_a - \sum_{(b,k) \in H} k N^{(b)}_k \alpha_b = \lambda \,\}. \tag{1.13}$$

In particular, (1.11) holds for any X_n of classical type.

The following fundamental conjecture is due to Kirillov and Reshetikhin:

Conjecture 1.8 ([K3, KR]). *(i) For any X_n of classical type, $\mathrm{ch}\, W^{(a)}_m$ equals to the RHSs of (1.7)–(1.10), respectively.*

(ii) For any X_n, $\mathrm{ch}\, W^{(a)}_m$ equals to the RHS of (1.11).

(iii) For any X_n, $\mathrm{ch}\, W^{(a)}_m$'s satisfy the relation (Q-I) with $Q^{(a)}_m$ being replaced with $\mathrm{ch}\, W^{(a)}_m$.

Remark 1.9. Precisely speaking, the existence of such modules $W^{(a)}_m(u)$ was claimed in [K3, KR] without the identification of their Drinfeld polynomials (1.1). According to Theorems 1.6 and 1.7, for any X_n of classical type, (i) and (ii) are equivalent and (iii) follows from (i). So far, Conjecture 1.8 has been completely proved in the literature only for A_n [K2] and D_n [Ch].

Since we expect that $Q^{(a)}_m$ are, in fact, $\mathcal{W}$ invariant also for any X_n of exceptional type, we reformulate Conjecture 1.8 more simply as

Conjecture 1.10. *For any X_n,*

$$\operatorname{ch} W_m^{(a)} = Q_m^{(a)}.$$

1.4. **Formal completeness of the XXX-type Bethe vectors.** Let us explain an interpretation of the expression (1.11) in the context of spin chains. Let

$$\mathcal{N} = \{\, N = (N_m^{(a)})_{(a,m)\in H} \mid N_m^{(a)} \in \mathbb{Z}_{\geq 0},\ \sum_{(a,m)\in H} N_m^{(a)} < \infty \,\}. \tag{1.14}$$

For each $\nu = (\nu_m^{(a)}) \in \mathcal{N}$, we associate a (finite) tensor product of $W_m^{(a)}$'s,

$$W^\nu = \bigotimes_{(a,m)\in H} (W_m^{(a)})^{\otimes \nu_m^{(a)}}. \tag{1.15}$$

In the context of the spin chain, W^ν appears as the *quantum space* on which the commuting family of the transfer matrices act. Thus, we call ν the *quantum space data*. For each $\nu \in \mathcal{N}$, we define a Laurent series $Q^\nu(x)$ of x,

$$Q^\nu(x) = \prod_{(a,m)\in H} (Q_m^{(a)}(x))^{\nu_m^{(a)}} \tag{1.16}$$

and expand $Q^\nu(x)$ as

$$Q^\nu(x) \prod_{\alpha\in\Delta_+} (1 - e^{-\alpha}) = \sum_{\lambda\in P} k_\lambda^\nu e^\lambda, \tag{1.17}$$

where Δ_+ is the set of all the positive roots of X_n, and P is the weight lattice of X_n. If Conjecture 1.10 is correct, then $Q^\nu = \operatorname{ch} W^\nu$. It follows from the Weyl character formula that k_λ^ν (for dominant X_n-weight λ) is equal to the multiplicity $[W^\nu : V_\lambda]$ of the $U_q(X_n)$-irreducible components V_λ with highest weight λ in W^ν. On the other hand, it was shown in [K2, HKOTY] (cf. (5.19)) that

$$k_\lambda^\nu = \sum_{N\in\mathcal{N}_\lambda^\nu} K(\nu, N) \tag{1.18}$$

holds under the assumption of the $\mathcal{W}$ invariance for $Q_m^{(a)}$. Here $K(\nu, N)$ and $\mathcal{N}_\lambda^\nu$ are defined in (4.6) and (5.15). The RHS of (1.18) represents a formal counting of the Bethe vectors of weight λ for the XXX-type ($Y(X_n)$) spin chain with quantum space W^ν [K1, K2, KR]. We say the counting *formal* because $K(\nu, N)$ correctly counts the Bethe vectors only for special (ν, N)'s. Therefore, we call the equality (1.18) the *formal completeness* of the XXX-type Bethe vectors (Corollary 5.11). The formula (1.11) is the special case of (1.18) for $\nu = (\nu_k^{(b)})$, $\nu_k^{(b)} = \delta_{ab}\delta_{mk}$.

1.5. **Formal completeness of the XXZ-type Bethe vectors.** We remind that the most significant difference between the Bethe vectors of the spin chains of XXX-type and XXZ-type is that the former are X_n-singular, while the latter are not necessarily $U_q(X_n)$-singular. Accordingly, we expand $Q^\nu(x)$ as

$$Q^\nu(x) = \sum_{\lambda\in P} r_\lambda^\nu e^\lambda. \tag{1.19}$$

If Conjecture 1.10 is correct, then r_λ^ν (for any X_n-weight λ) is equal to the weight multiplicity $\dim W_\lambda^\nu$ in W^ν at λ. We will show that (cf. (5.14))

$$r_\lambda^\nu = \sum_{N\in\mathcal{N}_\lambda^\nu} R(\nu, N), \tag{1.20}$$

where $R(\nu, N)$ is the integer defined in (3.2). It will be further shown that the RHS of (1.20) now represents a formal counting of the Bethe vectors of weight λ for the XXZ-type ($U_q(X_n^{(1)})$) spin chain with quantum space W^ν in the $q \to 0$ limit (Theorems 2.11 and 3.2). Therefore, we call the equality (1.20) the formal completeness of the XXZ-type Bethe vectors (Corollary 5.6). As the special case of (1.20) for $\nu = (\nu_k^{(b)})$, $\nu_k^{(b)} = \delta_{ab}\delta_{mk}$, we obtain yet another expression of $Q_m^{(a)}(x)$ (Theorem 5.3):

$$Q_m^{(a)}(x) = \sum_{\lambda \in P} \left\{ \sum_{N \in \mathcal{N}_\lambda} \left(\det_{(b,k),(c,j)} F_{bk,cj} \right) \prod_{(b,k)} \frac{1}{N_k^{(b)}} \binom{P_k^{(b)} + N_k^{(b)} - 1}{N_k^{(b)} - 1} \right\} e^\lambda, \tag{1.21}$$

$$F_{bk,cj} = \delta_{bc}\delta_{kj} P_k^{(b)} + (\alpha_b|\alpha_c) \min(t_c k, t_b j) N_j^{(c)}, \tag{1.22}$$

where $P_k^{(b)}$ and $\mathcal{N}_\lambda$ are in (1.12) and (1.13), and det and $\prod$ mean the ones over the index set $\{ (b,k) \in H \mid N_k^{(b)} > 0 \}$. The equality (1.21) holds for any X_n without assuming the $\mathcal{W}$ invariance of $Q_m^{(a)}(x)$.

The paper essentially consists of two parts: In the first part (Sections 2 and 3), we derive the number $R(\nu, N)$ in (1.20) from a formal counting of the XXZ-type Bethe vectors in the $q \to 0$ limit. In Theorem 2.11 we show that there is a one-to-one correspondence between a class of solutions of the Bethe equation and the one of the associated linear congruence equation called the string center equation (SCE). We then apply the standard Möbius inversion technique to count the off-diagonal solutions of the SCE (Theorem 3.2). In the second part (Sections 4 and 5), we show the formal completeness of the XXZ-type Bethe vectors (1.20). For that purpose we introduce the generating series of the numbers $R(\nu, N)$ and derive its analytic expression (Theorem 4.7). We then show the uniqueness of the series $\tilde{Q}_m^{(a)}$ and that a specialization of the above generating series indeed satisfies the condition in Definition 1.3 (Theorem 5.3). The two parts are logically almost independent and therefore able to be read rather separately.

2. Bethe equation at $q = 0$

2.1. **The $U_q(X_n^{(1)})$ Bethe equation.** Let $\mathcal{N}$ be the set in (1.14). Given $\nu \in \mathcal{N}$ and a sequence of n non-negative integers $(M_a)_{a=1}^n$, we associate a system of $\sum_{a=1}^n M_a$ equations for $\sum_{a=1}^n M_a$ variables $u_i^{(a)}$ ($a = 1, \dots, n; i = 1 \dots, M_a$),

$$\begin{aligned} &\prod_{m=1}^{\infty} \left(\frac{\sin \pi\Big(u_i^{(a)} + \frac{\sqrt{-1}m\hbar}{t_a}\Big)}{\sin \pi\Big(u_i^{(a)} - \frac{\sqrt{-1}m\hbar}{t_a}\Big)} \right)^{\nu_m^{(a)}} \\ &= - \prod_{b=1}^{n} \prod_{j=1}^{M_b} \frac{\sin \pi\Big(u_i^{(a)} - u_j^{(b)} + \sqrt{-1}(\alpha_a|\alpha_b)\hbar\Big)}{\sin \pi\Big(u_i^{(a)} - u_j^{(b)} - \sqrt{-1}(\alpha_a|\alpha_b)\hbar\Big)} \end{aligned} \tag{2.1}$$

with t_a and α_a defined in Section 1.2. We call (2.1) the *Bethe (ansatz) equation*. The equation was introduced in [OW, RW]. It is widely believed that for each

solution of (2.1), one can associate a vector in the space W^ν in (1.15) with X_n-weight

$$\sum_{(a,m)\in H} m\nu_m^{(a)}\Lambda_a - \sum_{a=1}^{n} M_a\alpha_a, \tag{2.2}$$

and the vector is an eigenvector (the *Bethe vector*) of the transfer matrix of the $U_q(X_n^{(1)})$ spin chain with the quantum space W^ν, if it is not the zero vector.

Remark 2.1. Actually, the equation (2.1) is a special case of a more general family of the Bethe equations which depend on the spectral parameters at each site of the spin chain. Since the analysis below can be easily extended for a general case in a straightforward way, we concentrate on the homogeneous case (2.1).

By setting $t = \max_{1\le a\le n} t_a$, $q = e^{-2\pi\hbar/t}$, $x_i^{(a)} = e^{2\pi\sqrt{-1}u_i^{(a)}}$, (2.1) is written as

$$F_{i+}^{(a)} G_{i-}^{(a)} = -F_{i-}^{(a)} G_{i+}^{(a)}, \tag{2.3}$$

$$F_{i+}^{(a)} = \prod_{m=1}^{\infty} (x_i^{(a)} q^{mt/t_a} - 1)^{\nu_m^{(a)}}, \quad G_{i+}^{(a)} = \prod_{b=1}^{n}\prod_{j=1}^{M_b} (x_i^{(a)} q^{(\alpha_a|\alpha_b)t} - x_j^{(b)}), \tag{2.4}$$

$$F_{i-}^{(a)} = \prod_{m=1}^{\infty} (x_i^{(a)} - q^{mt/t_a})^{\nu_m^{(a)}}, \quad G_{i-}^{(a)} = \prod_{b=1}^{n}\prod_{j=1}^{M_b} (x_i^{(a)} - x_j^{(b)} q^{(\alpha_a|\alpha_b)t}). \tag{2.5}$$

2.2. **String solution.** We consider a class of solutions $(x_i^{(a)})$ of (2.3) such that $x_i^{(a)} = x_i^{(a)}(q)$ is meromorphic (with respect to q) around $q = 0$. For a meromorphic function $f(q)$ around $q = 0$, let $\mathrm{ord}(f)$ be the order of the leading power of the Laurent expansion of $f(q)$ around $q = 0$, i.e.,

$$f(q) = q^{\mathrm{ord}(f)}(f^0 + f^1 q + \cdots), \qquad f^0 \neq 0, \tag{2.6}$$

and let $\tilde{f}(q) := f^0 + f^1 q + \cdots$ be the normalized series. When $f(q)$ is identically zero, we set $\mathrm{ord}(f) = \infty$.

Definition 2.2. A meromorphic solution $(x_i^{(a)})$ of (2.3) around $q = 0$ is called *admissible* (*inadmissible*) if $\mathrm{ord}(F_{i+}^{(a)} G_{i-}^{(a)}) < \infty$ for any (a, i) (otherwise).

For each $N = (N_m^{(a)}) \in \mathcal{N}$, we set

$$H' = H'(N) := \{\ (a,m) \in H \mid N_m^{(a)} > 0\ \}, \tag{2.7}$$

where H is defined in (1.2). We have $|H'| < \infty$.

Definition 2.3. Let $(M_a)_{a=1}^n$ be the one in the Bethe equation (2.3), and let $N = (N_m^{(a)}) \in \mathcal{N}$ satisfy $\sum_{m=1}^{\infty} mN_m^{(a)} = M_a$. A meromorphic solution $(x_i^{(a)})$ of (2.3) around $q = 0$ is called a *string solution of pattern* N if

(i) $(x_i^{(a)})$ is admissible.

(ii) $(x_i^{(a)})$ can be arranged as $(x_{m\alpha i}^{(a)})$ with

$$(a,m) \in H', \quad \alpha = 1, \dots, N_m^{(a)}, \quad i = 1, \dots, m \tag{2.8}$$

such that

(a) $d_{m\alpha i}^{(a)} := \mathrm{ord}(x_{m\alpha i}^{(a)}) = (m + 1 - 2i)t/t_a$.

(b) $z^{(a)}_{m\alpha} := x^{(a)0}_{m\alpha 1} = x^{(a)0}_{m\alpha 2} = \cdots = x^{(a)0}_{m\alpha m}$ ($\neq 0$), where $x^{(a)0}_{m\alpha i}$ is the coefficient of the leading power of $x^{(a)}_{m\alpha i}$.

For each (a, m, α), $(x^{(a)}_{m\alpha i})^m_{i=1}$ is called an *m-string of color a*, and $z^{(a)}_{m\alpha}$ is called the *string center* of the m-string $(x^{(a)}_{m\alpha i})^m_{i=1}$. (Thus, $N^{(a)}_m$ is the number of the m-strings of color a.)

For a string solution $(x^{(a)}_{m\alpha i})$, $x^{(a)}_{m\alpha i}(q) = q^{d^{(a)}_{m\alpha i}}\tilde{x}^{(a)}_{m\alpha i}(q)$, of pattern N, the Bethe equation (2.3) reads

$$F^{(a)}_{m\alpha i+}G^{(a)}_{m\alpha i-} = -F^{(a)}_{m\alpha i-}G^{(a)}_{m\alpha i+}, \tag{2.9}$$

$$F^{(a)}_{m\alpha i+} = \prod_{k=1}^{\infty}(\tilde{x}^{(a)}_{m\alpha i}q^{d^{(a)}_{m\alpha i}+kt/t_a} - 1)^{\nu^{(a)}_k}, \tag{2.10}$$

$$F^{(a)}_{m\alpha i-} = \prod_{k=1}^{\infty}(\tilde{x}^{(a)}_{m\alpha i}q^{d^{(a)}_{m\alpha i}} - q^{kt/t_a})^{\nu^{(a)}_k}, \tag{2.11}$$

$$G^{(a)}_{m\alpha i+} = \prod_{bk\beta j}(\tilde{x}^{(a)}_{m\alpha i}q^{d^{(a)}_{m\alpha i}+(\alpha_a|\alpha_b)t} - \tilde{x}^{(b)}_{k\beta j}q^{d^{(b)}_{k\beta j}}), \tag{2.12}$$

$$G^{(a)}_{m\alpha i-} = \prod_{bk\beta j}(\tilde{x}^{(a)}_{m\alpha i}q^{d^{(a)}_{m\alpha i}} - \tilde{x}^{(b)}_{k\beta j}q^{d^{(b)}_{k\beta j}+(\alpha_a|\alpha_b)t}), \tag{2.13}$$

where $\prod_{bk\beta j} = \prod_{(b,k)\in H'}\prod_{\beta=1}^{N^{(b)}_k}\prod_{j=1}^{k}$, and the indices a, m, α, and i run in the range (2.8). For a string solution $(x^{(a)}_{m\alpha i})$ of type N, we call (2.9) also the Bethe equation.

Notice that $\zeta^{(a)}_{m\alpha i} := \mathrm{ord}(\tilde{x}^{(a)}_{m\alpha i} - \tilde{x}^{(a)}_{m\alpha i-1})$ is positive and finite because of (i) and (ii) in Definition 2.3. We define $y^{(a)}_{m\alpha i}(q)$ $(2 \leq i \leq m)$ as

$$q^{\zeta^{(a)}_{m\alpha i}}y^{(a)}_{m\alpha i}(q) = \tilde{x}^{(a)}_{m\alpha i}(q) - \tilde{x}^{(a)}_{m\alpha i-1}(q). \tag{2.14}$$

Let us extract the factors $y^{(a)}_{m\alpha i}$ from $G^{(a)}_{m\alpha i\pm}$ and introduce $G'^{(a)}_{m\alpha i\pm}$ as follows:

$$G^{(a)}_{m\alpha i+} = \begin{cases} G'^{(a)}_{m\alpha 1+} & i = 1 \\ G'^{(a)}_{m\alpha i+}q^{d^{(a)}_{m\alpha i}+(\alpha_a|\alpha_a)t+\zeta^{(a)}_{m\alpha i}}y^{(a)}_{m\alpha i} & 2 \leq i \leq m, \end{cases} \tag{2.15}$$

$$G^{(a)}_{m\alpha i-} = \begin{cases} G'^{(a)}_{m\alpha i-}q^{d^{(a)}_{m\alpha i}+\zeta^{(a)}_{m\alpha i+1}}y_{m\alpha i+1} & 1 \leq i \leq m-1 \\ G'^{(a)}_{m\alpha m-} & i = m. \end{cases} \tag{2.16}$$

Now the Bethe equation (2.9) takes the form:

$$\tilde{F}^{(a)}_{m\alpha 1+}\tilde{G}'^{(a)}_{m\alpha 1-}y^{(a)}_{m\alpha 2} = -\tilde{F}^{(a)}_{m\alpha 1-}\tilde{G}'^{(a)}_{m\alpha 1+} \qquad i = 1, \tag{2.17}$$

$$\tilde{F}^{(a)}_{m\alpha i+}\tilde{G}'^{(a)}_{m\alpha i-}y^{(a)}_{m\alpha i+1} = -\tilde{F}^{(a)}_{m\alpha i-}\tilde{G}'^{(a)}_{m\alpha i+}y^{(a)}_{m\alpha i} \qquad 2 \leq i \leq m-1, \tag{2.18}$$

$$\tilde{F}^{(a)}_{m\alpha m+}\tilde{G}'^{(a)}_{m\alpha m-} = -\tilde{F}^{(a)}_{m\alpha m-}\tilde{G}'^{(a)}_{m\alpha m+}y^{(a)}_{m\alpha m} \qquad i = m. \tag{2.19}$$

2.3. $q \to 0$ **limit of Bethe equation.** Suppose that $(x^{(a)}_{m\alpha i})$ is a string solution to the Bethe equation (2.9). Taking the leading coefficients of (2.17)–(2.19),

$$F^{(a)0}_{m\alpha 1+}G'^{(a)0}_{m\alpha 1-}y^{(a)0}_{m\alpha 2} = -F^{(a)0}_{m\alpha 1-}G'^{(a)0}_{m\alpha 1+} \qquad i=1, \tag{2.20}$$

$$F^{(a)0}_{m\alpha i+}G'^{(a)0}_{m\alpha i-}y^{(a)0}_{m\alpha i+1} = -F^{(a)0}_{m\alpha i-}G'^{(a)0}_{m\alpha i+}y^{(a)0}_{m\alpha i} \qquad 2 \le i \le m-1, \tag{2.21}$$

$$F^{(a)0}_{m\alpha m+}G'^{(a)0}_{m\alpha m-} = -F^{(a)0}_{m\alpha m-}G'^{(a)0}_{m\alpha m+}y^{(a)0}_{m\alpha m} \qquad i=m. \tag{2.22}$$

In particular, taking the product of (2.20)–(2.22), we have

$$1 = (-1)^m \prod_{i=1}^m \frac{F^{(a)0}_{m\alpha i+}G^{(a)0}_{m\alpha i-}}{F^{(a)0}_{m\alpha i-}G^{(a)0}_{m\alpha i+}}, \tag{2.23}$$

which turns out to be a key equation.

2.4. **Generic string solution.** We introduce a class of string solutions for which (2.23) can be explicitly written down. Let $\xi^{(a)}_{m\alpha i\pm}$, $\eta^{(a)}_{m\alpha i\pm}$ be the "superficial" orders of the factors $F^{(a)}_{m\alpha i\pm}$, $G^{(a)}_{m\alpha i\pm}$ in the Bethe equation (2.9): Namely,

$$\xi^{(a)}_{m\alpha i+} = \frac{t}{t_a}\sum_{k=1}^\infty \nu^{(a)}_k \min(m+1-2i+k, 0), \tag{2.24}$$

$$\xi^{(a)}_{m\alpha i-} = \frac{t}{t_a}\sum_{k=1}^\infty \nu^{(a)}_k \min(m+1-2i, k), \tag{2.25}$$

$$\eta^{(a)}_{m\alpha i+} = \frac{t}{t_a}\sum_{bk\beta j}\frac{1}{t_b}\min(t_b(m+1-2i)+t_at_b(\alpha_a|\alpha_b), t_a(k+1-2j)), \tag{2.26}$$

$$\eta^{(a)}_{m\alpha i-} = \frac{t}{t_a}\sum_{bk\beta j}\frac{1}{t_b}\min(t_b(m+1-2i), t_a(k+1-2j)+t_at_b(\alpha_a|\alpha_b)). \tag{2.27}$$

Definition 2.4. A string solution $(x^{(a)}_{m\alpha i})$ to (2.9) is called *generic* if

$$\begin{aligned} &\mathrm{ord}(F^{(a)}_{m\alpha i\pm}) = \xi^{(a)}_{m\alpha i\pm},\\ &\mathrm{ord}(G^{(a)}_{m\alpha i+}) = \eta^{(a)}_{m\alpha i+}+\zeta^{(a)}_{m\alpha i}, \qquad \mathrm{ord}(G^{(a)}_{m\alpha i-}) = \eta^{(a)}_{m\alpha i-}+\zeta^{(a)}_{m\alpha i+1}, \end{aligned} \tag{2.28}$$

where $\zeta^{(a)}_{m\alpha 1} = \zeta^{(a)}_{m\alpha m+1} = 0$.

Given a quantum space data $\nu \in \mathcal{N}$ and a string pattern $N \in \mathcal{N}$, we put

$$\gamma^{(a)}_m = \gamma^{(a)}_m(\nu) = \sum_{k=1}^\infty \min(m,k)\nu^{(a)}_k, \tag{2.29}$$

$$P^{(a)}_m = P^{(a)}_m(\nu, N) = \gamma^{(a)}_m - \sum_{(b,k)\in H}(\alpha_a|\alpha_b)\min(t_bm, t_ak)N^{(b)}_k. \tag{2.30}$$

Lemma 2.5. *We have*

$$\begin{aligned} &(\xi^{(a)}_{m\alpha i+}+\eta^{(a)}_{m\alpha i-}) - (\xi^{(a)}_{m\alpha i-}+\eta^{(a)}_{m\alpha i+})\\ &= \begin{cases} -\frac{t}{t_a}\left(P^{(a)}_{m+1-2i}+N^{(a)}_{m+1-2i}\right) - \Delta^{(a)}_{m+1-2i} & 1 \le i < \frac12(m+1)\\ 0 & i = \frac12(m+1)\\ \frac{t}{t_a}\left(P^{(a)}_{2i-m-1}+N^{(a)}_{2i-m-1}\right) + \Delta^{(a)}_{2i-m-1} & \frac12(m+1) < i \le m, \end{cases} \end{aligned} \tag{2.31}$$

where $\Delta^{(a)}_j = 0$ *except for the following cases: If* $t_a = 1$ *and there is* a' *such that* $t_{a'} \neq 1$, $C_{aa'} \neq 0$, *then*

$$\Delta^{(a)}_j = \begin{cases} -N^{(a')}_{2j} & t_{a'} = 2 \\ -\left(N^{(a')}_{3j-1} + 2N^{(a')}_{3j} + N^{(a')}_{3j+1}\right) & t_{a'} = 3. \end{cases} \tag{2.32}$$

Proof. It is easy to show them by the case check. □

Proposition 2.6. *A necessary condition for the existence of a generic string solution of pattern* N *is as follows (* $(a,m) \in H'$, $1 \le \alpha \le N^{(a)}_m$, $2 \le i \le m$*):*

$$\sum_{k=1}^{\min(i-1,m+1-i)} \left\{ \frac{t}{t_a}\left(P^{(a)}_{m+1-2k} + N^{(a)}_{m+1-2k}\right) + \Delta^{(a)}_{m+1-2k} \right\} > 0. \tag{2.33}$$

Proof. Suppose the equation (2.9) admits a generic solution. Then, $\xi^{(a)}_{m\alpha i+} + \eta^{(a)}_{m\alpha i-} + \zeta^{(a)}_{m\alpha i+1} = \xi^{(a)}_{m\alpha i-} + \eta^{(a)}_{m\alpha i+} + \zeta^{(a)}_{m\alpha i}$ holds. Therefore, the LHS of (2.31) equals to $\zeta^{(a)}_{m\alpha i} - \zeta^{(a)}_{m\alpha i+1}$. Then, the LHS of (2.33) equals to $\zeta^{(a)}_{m\alpha i}$, which is positive. □

2.5. **String center equation (SCE).** Let us compute the equation (2.23) for a *generic* string solution.

Proposition 2.7. *Let* $(x^{(a)}_{m\alpha i})$ *be a generic string solution of pattern* N*. Then its string centers* $(z^{(a)}_{m\alpha})$ *satisfy the following equations* ($(a,m) \in H'$, $1 \le \alpha \le N^{(a)}_m$)*:*

$$\prod_{(b,k)\in H'} \prod_{\beta=1}^{N^{(b)}_k} (z^{(b)}_{k\beta})^{A_{am\alpha,bk\beta}} = (-1)^{P^{(a)}_m + N^{(a)}_m + 1}, \tag{2.34}$$

$$A_{am\alpha,bk\beta} := \delta_{ab}\delta_{mk}\delta_{\alpha\beta}(P^{(a)}_m + N^{(a)}_m) + (\alpha_a|\alpha_b)\min(t_b m, t_a k) - \delta_{ab}\delta_{mk}. \tag{2.35}$$

We call (2.34) the *string center equation* (SCE) of pattern N. The SCE (2.34) becomes the linear congruence equation (also called the SCE) in terms of the variables $u_{k\beta}$ (*modulo* $\mathbb{Z}$) defined by $z^{(b)}_{k\beta} = \exp(2\pi\sqrt{-1}u^{(b)}_{k\beta})$:

$$\sum_{(b,k)\in H'} \sum_{\beta=1}^{N^{(b)}_k} A_{am\alpha,bk\beta} u^{(b)}_{k\beta} \equiv \frac{P^{(a)}_m + N^{(a)}_m + 1}{2} \mod \mathbb{Z}. \tag{2.36}$$

Proof. Let us compute the ratio (2.23) explicitly.

$$\prod_{i=1}^{m} F^{(a)0}_{m\alpha i\epsilon} = \begin{cases} (-1)^{\gamma^{(a)}_m} \prod_{k=1}^{\infty} (f^k_{am\alpha})^{\nu^{(a)}_k} & \epsilon = + \\ (z^{(a)}_{m\alpha})^{\gamma^{(a)}_m} \prod_{k=1}^{\infty} (f^k_{am\alpha})^{\nu^{(a)}_k} & \epsilon = -, \end{cases}$$

$$f^k_{am\alpha} = \begin{cases} 1 & m \le k \\ (-z^{(a)}_{m\alpha})^{(m-k)/2} & m > k,\ k \equiv m \bmod 2 \\ (-z^{(a)}_{m\alpha})^{(m-k-1)/2}(z^{(a)}_{m\alpha} - 1) & m > k,\ k \not\equiv m \bmod 2. \end{cases}$$

In order to calculate $\prod_{i=1}^m (G^{(a)0}_{m\alpha i-}/G^{(a)0}_{m\alpha i+})$, it is convenient to evaluate

$$\prod_{i=1}^{m}\prod_{j=1}^{k}(\tilde{x}^{(a)}_{m\alpha i}q^{d^{(a)}_{m\alpha i}+(1+\epsilon)(\alpha_a|\alpha_b)t/2} - \tilde{x}^{(b)}_{k\beta j}q^{d^{(b)}_{k\beta j}+(1-\epsilon)(\alpha_a|\alpha_b)t/2})^0$$
$$= \begin{cases} (-z^{(b)}_{k\beta})^{(\alpha_a|\alpha_b)\min(t_b m, t_a k)-\delta_{ab}\delta_{mk}} g^{bk\beta}_{am\alpha} & \epsilon = 1 \\ (z^{(a)}_{m\alpha})^{(\alpha_a|\alpha_b)\min(t_b m, t_a k)-\delta_{ab}\delta_{mk}} (-1)^{(m-1)\delta_{ab}\delta_{mk}\delta_{\alpha\beta}} g^{bk\beta}_{am\alpha} & \epsilon = -1. \end{cases}$$

Here $g^{bk\beta}_{am\alpha}$ are given as follows:

(i) For $t_b(m-1) - t_a(k-1) - t_a t_b(\alpha_a|\alpha_b) \equiv 1 \bmod 2$

$$g^{bk\beta}_{am\alpha} = (-z^{(a)}_{m\alpha}z^{(b)}_{k\beta})^{\frac{1}{2}mk - \frac{1}{2}(\alpha_a|\alpha_b)\min(t_b m, t_a k)}.$$

(ii) For $t_b(m-1) - t_a(k-1) - t_a t_b(\alpha_a|\alpha_b) \equiv 0 \bmod 2$, $(a,m,\alpha) \neq (b,k,\beta)$

$$g^{bk\beta}_{am\alpha} = (-z^{(a)}_{m\alpha}z^{(b)}_{k\beta})^{\frac{1}{2}mk - \left(\frac{1}{2t_{ab}} + \frac{1}{2}(\alpha_a|\alpha_b)\right)\left(\min(t_b m, t_a k) + \Delta^{bk}_{am}\right) + \delta_{ab}\delta_{mk}}$$
$$\times (z^{(a)}_{m\alpha} - z^{(b)}_{k\beta})^{\frac{1}{t_{ab}}\left(\min(t_b m, t_a k) + (1-t)\Delta^{bk}_{am}\right) - \delta_{ab}\delta_{mk}},$$

where $t_{ab} = \max(t_a, t_b)$, and Δ^{bk}_{am} is 0 except for the cases: (a) If $t_a < t_b$, $t_b m > t_a k$, $t_b m - t_a k \equiv \pm 1 \bmod 2t$, then Δ^{bk}_{am} is 1 or -1 with $\Delta^{bk}_{am} \equiv t_b m - t_a k \bmod 2t$. (b) If $t_a > t_b$, $t_b m < t_a k$, $t_b m - t_a k \equiv \pm 1 \bmod 2t$, then Δ^{bk}_{am} is 1 or -1 with $-\Delta^{bk}_{am} \equiv t_b m - t_a k \bmod 2t$.

(iii) For $(a,m,\alpha) = (b,k,\beta)$,

$$g^{bk\beta}_{am\alpha} = (-(z^{(a)}_{m\alpha})^2)^{\frac{1}{2}m^2 - \frac{3}{2}m + 1} y^{(a)0}_{m\alpha 2} \cdots y^{(a)0}_{m\alpha m}.$$

The factors $(f^k_{am\alpha})^{\nu^{(a)}_k}$ and $g^{bk\beta}_{am\alpha}$ are all nonzero for a generic string solution $(x^{(a)}_{m\alpha i})$. Thus they are canceled in the ratio in the RHS of (2.23), and we find

$$(-1)^m \prod_{i=1}^{m} \frac{F^{(a)0}_{m\alpha i+}G^{(a)0}_{m\alpha i-}}{F^{(a)0}_{m\alpha i-}G^{(a)0}_{m\alpha i+}} = (-1)^{P^{(a)}_m + N^{(a)}_m + 1} \prod_{bk\beta} (z^{(b)}_{k\beta})^{-A_{am\alpha, bk\beta}}. \tag{2.37}$$

From (2.23) and (2.37) we obtain (2.34). □

From the conditions $(f^k_{am\alpha})^{\nu^{(a)}_k} \neq 0$ and $g^{bk\beta}_{am\alpha} \neq 0$ in the above proof, we see that a string solution is generic if and only if its string centers $(z^{(a)}_{m\alpha})$ satisfy the following condition for any (a,m,α):

$$\begin{gathered} \prod_{\substack{k=1 \\ k \not\equiv m\,(2)}}^{m-1} (z^{(a)}_{m\alpha} - 1)^{\nu^{(a)}_k} \neq 0, \\ \prod_{\substack{bk\beta\,(\neq am\alpha) \\ t_b(m-1) - t_a(k-1) \\ -t_a t_b(\alpha_a|\alpha_b) \equiv 0\,(2)}} (z^{(a)}_{m\alpha} - z^{(b)}_{k\beta})^{\frac{1}{t_{ab}}\left(\min(t_b m, t_a k) + (1-t)\Delta^{bk}_{am}\right) - \delta_{ab}\delta_{mk}} \neq 0. \end{gathered} \tag{2.38}$$

Definition 2.8. A solution to the SCE (2.34) is called *generic* if it satisfies the condition (2.38).

Let A be the matrix with the entry $A_{am\alpha,bk\beta}$ in (2.35).

Proposition 2.9. *Suppose that* $N \in \mathcal{N}$ *satisfies the conditions (2.33) and* $\det A \neq 0$. *Then, for each generic solution* $(z^{(a)}_{m\alpha})$ *to the SCE of pattern* N*, there exists a unique generic string solution* $(x'^{(a)}_{m\alpha i}(q))$ *of pattern* N *to the Bethe equation (2.9) such that its string center* $z'^{(a)}_{m\alpha}$ *is equal to* $z^{(a)}_{m\alpha}$.

To prove Proposition 2.9, we introduce new variables $w^{(a)}_{m\alpha i}$ as

$$w^{(a)}_{m\alpha i} = \begin{cases} \tilde{x}^{(a)}_{m\alpha i} & i = 1 \\ y^{(a)}_{m\alpha i} & 2 \leq i \leq m. \end{cases} \tag{2.39}$$

Then

$$\tilde{x}^{(a)}_{m\alpha i} = w^{(a)}_{m\alpha 1} + q^{\zeta^{(a)}_{m\alpha 2}} w^{(a)}_{m\alpha 2} + \cdots + q^{\zeta^{(a)}_{m\alpha i}} w^{(a)}_{m\alpha i} \qquad 1 \leq i \leq m. \tag{2.40}$$

Let us write the ith equation of (2.17)–(2.19) as $L^{(a)}_{m\alpha i} = R^{(a)}_{m\alpha i}$. Let $J = (J_{am\alpha i, bk\beta j})$ be a matrix with entry $J_{am\alpha i, bk\beta j} = \frac{\partial}{\partial w^{(b)}_{k\beta j}} \big(\frac{L^{(a)}_{m\alpha i}}{R^{(a)}_{m\alpha i}} - 1 \big)$.

Lemma 2.10. *If* $N \in \mathcal{N}$ *satisfies the conditions (2.33) and* $\det A \neq 0$*, then* $\det J$ *is not zero at* $q = 0$.

Proof. Owing to the assumption (2.33), we have $\zeta^{(a)}_{m\alpha i} > 0$ for any (m, α, i). Since $L^{(a)0}_{m\alpha i} = R^{(a)0}_{m\alpha i} \neq 0$, it suffices to show $\det \mathcal{J} \neq 0$ for $\mathcal{J}_{am\alpha i, bk\beta j} = \frac{\partial}{\partial w^{(b)}_{k\beta j}} \log \frac{L^{(a)}_{m\alpha i}}{R^{(a)}_{m\alpha i}}$. From (2.40) both $\frac{\partial \tilde{F}^{(a)}_{m\alpha i \pm}}{\partial w^{(b)}_{k\beta j}}$ and $\frac{\partial \tilde{G}'^{(a)}_{m\alpha i \pm}}{\partial w^{(b)}_{k\beta j}}$ for $j \neq 1$ are zero at $q = 0$. Thus among $\mathcal{J}^0_{am\alpha i, bk\beta j}$'s the non-vanishing ones are only $\mathcal{J}^0_{am\alpha i, bk\beta 1}$ $(1 \leq i \leq m)$, $\mathcal{J}^0_{am\alpha i, am\alpha i} = -1/y^{(a)0}_{m\alpha i}$ $(2 \leq i \leq m)$, and $\mathcal{J}^0_{am\alpha i, am\alpha i+1} = 1/y^{(a)0}_{m\alpha i+1}$ $(1 \leq i \leq m-1)$. Let $\vec{\mathcal{J}}^0_{am\alpha i} = (\mathcal{J}^0_{am\alpha i, bk\beta j})_{bk\beta j}$ be the $(am\alpha i)$-th row vector of the matrix $\mathcal{J}$. In view of the above result, the linear dependence $\sum_{am\alpha i} c_{am\alpha i} \vec{\mathcal{J}}^0_{am\alpha i} = 0$ can possibly hold only when $c_{am\alpha i}$ is independent of i. Consequently we consider the equation $\sum_{am\alpha} c_{am\alpha} \sum_{i=1}^m \vec{\mathcal{J}}^0_{am\alpha i} = 0$. The $(bk\beta 1)$-th component of the vector $\sum_{i=1}^m \vec{\mathcal{J}}^0_{am\alpha i}$ is given by

$$\lim_{q \to 0} \frac{\partial}{\partial w^{(b)}_{k\beta 1}} \log \prod_{i=1}^m \frac{-\tilde{F}^{(a)}_{m\alpha i+} \tilde{G}'^{(a)}_{m\alpha i-}}{\tilde{F}^{(a)}_{m\alpha i-} \tilde{G}'^{(a)}_{m\alpha i+}} = \frac{\partial}{\partial z^{(b)}_{k\beta}} \log \prod_{i=1}^m \frac{-F^{(a)0}_{m\alpha i+} G^{(a)0}_{m\alpha i-}}{F^{(a)0}_{m\alpha i-} G^{(a)0}_{m\alpha i+}},$$

where we have taken into account (2.40) and $\zeta^{(a)}_{m\alpha i} > 0$. Due to (2.37) the last expression is equal to $-A_{am\alpha, bk\beta}/z^{(b)}_{k\beta}$. Therefore the equation $\sum_{am\alpha} c_{am\alpha} \sum_{i=1}^m \vec{\mathcal{J}}^0_{am\alpha i} = 0$ is equivalent to $\sum_{am\alpha} c_{am\alpha} A_{am\alpha, bk\beta} = 0$ for any (b, k, β). This admits only the trivial solution for $c_{am\alpha}$ if $\det A \neq 0$. □

Proof of Proposition 2.9. The Bethe equations (2.17)–(2.19) are simultaneous equations in the variables $((w^{(a)}_{m\alpha i}), q)$. At $q = 0$, (2.17)–(2.19) reduce to (2.20)–(2.22). The latter fix $(y^{(a)0}_{m\alpha i})$ unambiguously once a generic solution $(z^{(a)}_{m\alpha})$ to the SCE is given. Denote the resulting value of $w^{(a)}_{m\alpha i}$ by $w^{(a)0}_{m\alpha i}$. From Lemma 2.10 and the implicit function theorem, there uniquely exist the functions $w'^{(a)}_{m\alpha i}(q)$ satisfying (2.17)–(2.19) and $w'^{(a)0}_{m\alpha i} = w^{(a)0}_{m\alpha i}$. □

From Propositions 2.7 and 2.9, we obtain the main statement in this section.

Theorem 2.11. *Suppose that $N \in \mathcal{N}$ satisfies the conditions (2.33) and* $\det A \neq 0$. *Then, there is a one-to-one correspondence between generic string solutions of pattern N to the Bethe equation (2.9) and generic solutions to the SCE (2.34) of pattern N.*

Remark 2.12. As we will see later in Lemma 3.8, the condition $\det A \neq 0$ in Theorem 2.11 is satisfied if $N \in \mathcal{N}$ satisfy the condition:

$$P_m^{(a)}(\nu, N) \geq 0 \text{ for any } (a, m) \in H'. \tag{2.41}$$

More strongly, $\det A$ is positive under (2.41). In general, the conditions (2.33) and (2.41) are simultaneously satisfied if $\sum_m mN_m^{(a)}$ is sufficiently smaller than $\sum_m \nu_m^{(a)}$. Because of (2.2), $\sum_{(a,m)\in H} mN_m^{(a)}\alpha_a$ measures the difference between the weight of the corresponding Bethe vector and the highest weight of the quantum space W^ν. Thus, if $\sum_m \nu_m^{(a)}$ are large enough, the conditions in Theorem 2.11 are satisfied at least "near the highest weight".

3. Counting of off-diagonal solutions to SCE

In this section the off-diagonal solutions of the SCE will be counted under a certain condition (Theorem 3.2).

3.1. **Off-diagonal solution.** In what follows, the symbol $\binom{k}{j}$ ($k \in \mathbb{C}$, $j \in \mathbb{Z}$) will denote the binomial coefficient:

$$\binom{k}{j} = \begin{cases} k(k-1)\cdots(k-j+1)/j! & j > 0 \\ 1 & j = 0 \\ 0 & j < 0. \end{cases} \tag{3.1}$$

For each ν, $N \in \mathcal{N}$, we define the number $R(\nu, N)$ as follows: For $N \neq 0 \in \mathcal{N}$,

$$R(\nu, N) = \Big(\det_{(a,m),(b,k)\in H'} F_{am,bk}\Big) \prod_{(a,m)\in H'} \frac{1}{N_m^{(a)}} \binom{P_m^{(a)} + N_m^{(a)} - 1}{N_m^{(a)} - 1}, \tag{3.2}$$

$$F_{am,bk} = \delta_{ab}\delta_{mk}P_m^{(a)} + (\alpha_a|\alpha_b)\min(t_b m, t_a k)N_k^{(b)}, \tag{3.3}$$

where $H' = H'(N)$ and $P_m^{(a)} = P_m^{(a)}(\nu, N)$ are given by (2.7) and (2.30). For $N = 0$, we set $R(\nu, 0) = 1$ irrespective of ν. For any $N \in \mathcal{N}$, $R(\nu, N)$ is an integer (cf. Lemma 4.2), though it is not always a positive one.

Definition 3.1. A solution $(z_{m\alpha}^{(a)})$ to the SCE is called *off-diagonal* (*diagonal*) if $z_{m\alpha}^{(a)} = z_{m\beta}^{(a)}$ only for $\alpha = \beta$ (otherwise).

Notice that "off-diagonal" above is weaker than the condition that $z_{m\alpha}^{(a)}$'s are *all distinct*.

Theorem 3.2. *If $N\,(\neq 0) \in \mathcal{N}$ satisfies the condition (2.41), then the number of off-diagonal solutions to the SCE (2.34) of pattern N divided by $\prod_{(a,m)\in H'} N_m^{(a)}!$ is equal to $R(\nu, N)$.*

Remark 3.3. In contrast with Definition 3.1, let us call a solution to the Bethe equation (2.9) *off-diagonal* if $x^{(a)}_{m\alpha i}(q)$'s are all distinct. Theorem 3.2 is motivated by the fact for $X_n = A_1$ [TV] that (i) the Bethe vector associated to each solution to the Bethe equation does not vanish if and only if $x^{(1)}_i$'s are admissible and off-diagonal; (ii) the Bethe vector is invariant under the permutations of $x^{(1)}_i$'s. Combining Theorem 3.2 with Theorem 2.11, the number $R(\nu, N)$ correctly counts the off-diagonal string solutions of pattern N (modulo permutation) to the Bethe equation (2.9) if N satisfies all the following conditions:

(i) $N \in \mathcal{N}$ satisfies the conditions (2.33) and (2.41).

(ii) All the off-diagonal string solutions of pattern N to the Bethe equation are generic.

(iii) For each off-diagonal string solution of pattern N to the Bethe equation, the string centers are all distinct.

(iv) For each off-diagonal solution to the SCE of pattern N, $z^{(a)}_{m\alpha}$'s are all distinct.

Unfortunately, so far we do not know a more general condition where the one-to-one correspondence between the off-diagonal string solutions to the Bethe equation (2.9) and the off-diagonal solutions to the SCE (2.34) holds.

3.2. **Proof of Theorem 3.2.** We work with the logarithmic form of the SCE (2.36),

$$A\vec{u} \equiv \vec{c} \mod \mathbb{Z}^d, \tag{3.4}$$

where $d = \sum_{(a,m)\in H'} N^{(a)}_m$. Here $\vec{u} = (u^{(b)}_{k\beta})$ is the unknown and $\vec{c}$ is some constant vector. The matrix $A = (A_{am\alpha,bk\beta})$ is specified by (2.35). We consider the solutions $u^{(b)}_{k\beta}$ *modulo* integers.

The following fact is well-known (cf. [C, 1.2.2, Lemma 1]):

Lemma 3.4. *Let B be an r by r integer matrix with $\det B \neq 0$. Then for any $\vec{b} \in \mathbb{R}^r$, the equation $B\vec{x} \equiv \vec{b} \bmod \mathbb{Z}^r$ has exactly $|\det B|$ solutions $\vec{x}$ in $(\mathbb{R}/\mathbb{Z})^r$.*

Therefore, if $\det A \neq 0$, then the number of the solutions to (3.4), including diagonal ones, is given by $|\det A|$. One can systematically remove the diagonal solutions using the Möbius inversion method [A, S] as follows.

For a given positive integer K, $\pi = (\pi_1, \dots, \pi_l)$ is called a *partition* of a set $\{1, \dots, K\}$ if

$$\{1, \dots, K\} = \pi_1 \sqcup \cdots \sqcup \pi_l$$

is a disjoint union decomposition. Here, the ordering of $\pi_1, \pi_2, \dots, \pi_l$ is ignored. Each π_i is called a *block* of π, and l is called a *length* of π. Let L_K denote the set of partitions of $\{1, \dots, K\}$. L_K becomes a partially ordered set (poset) by the following partial order: Given two partitions $\pi, \pi' \in L_K$, we say $\pi \leq \pi'$ if each block of π' is contained in a block of π. Let $\mu(\pi, \pi')$ be the Möbius function for the poset L_K. It is well-known [A, S] that

Lemma 3.5. *Let X be an indeterminate. For any $\pi \in L_K$ we have*

$$(X)_{l(\pi)} = \sum_{\pi' \leq \pi} \mu(\pi', \pi) X^{l(\pi')}, \tag{3.5}$$

where $(X)_l = X(X-1)\cdots(X-l+1)$.

For a given string pattern $N \in \mathcal{N}$, we consider the direct product of posets $\mathcal{L}_N = \prod_{(a,m)\in H'} L_{N_m^{(a)}}$; for $\pi = (\pi_m^{(a)})$, $\pi' = (\pi_m'^{(a)}) \in \mathcal{L}_N$, we define $\pi \le \pi'$ when $\pi_m^{(a)} \le \pi_m'^{(a)}$ for each (a,m). Below, $\mu(\pi,\pi')$ means the Möbius function for $\mathcal{L}_N$. We set $l(\pi) = \sum_{(a,m)\in H'} l(\pi_m^{(a)})$. For each $\pi = (\pi_m^{(a)}) \in \mathcal{L}_N$, let

$$\begin{aligned}\mathrm{Sol}'_\pi = \{\, \vec{u} = (u_{k\beta}^{(b)}) \mid {}& \vec{u} \text{ is a solution of (3.4)},\ u_{k\alpha}^{(b)} = u_{k\beta}^{(b)} \text{ if}\\ &\alpha \text{ and } \beta \text{ belong to the same block of } \pi_k^{(b)} \,\},\\ \mathrm{Sol}_\pi = \{\, \vec{u} = (u_{k\beta}^{(b)}) \mid {}& \vec{u} \text{ is a solution of (3.4)},\ u_{k\alpha}^{(b)} = u_{k\beta}^{(b)} \text{ if and only if}\\ &\alpha \text{ and } \beta \text{ belong to the same block of } \pi_k^{(b)} \,\}.\end{aligned}$$

In particular, $\mathrm{Sol}_{\pi_{\max}}$ is the set of the off-diagonal solutions of (3.4), where $\pi_{\max}$ is the maximal element in $\mathcal{L}_N$.

Lemma 3.6.

$$|\mathrm{Sol}_\pi| = \sum_{\pi' \le \pi} \mu(\pi',\pi)|\mathrm{Sol}'_{\pi'}| \tag{3.6}$$

Proof. By definition $|\mathrm{Sol}'_\pi| = \sum_{\pi'\le\pi} |\mathrm{Sol}_{\pi'}|$. Applying the Möbius inversion formula [S], we obtain (3.6). □

To use the formula (3.6), let us evaluate $|\mathrm{Sol}'_\pi|$. With the constraint, $u_{k\alpha}^{(b)} = u_{k\beta}^{(b)}$ on $\vec{u} = (u_{k\beta}^{(b)})$ if α and β belong to the same block of $\pi_k^{(b)}$, the SCE (3.4) reduces to the following form:

$$A^\pi \vec{u}_\pi \equiv \vec{c}_\pi \mod \mathbb{Z}^{l(\pi)} \tag{3.7}$$

In the new unknown $\vec{u}_\pi = (u_{k\beta}^{(b)})$, β is now labeled by the blocks of $\pi_k^{(b)}$. The matrix A^π is an integer matrix of size $l(\pi)$ obtained by a reduction of the matrix A as follows: It is formed by summing up the $(bk\beta)$-th columns of A over those β belonging to the same block of $\pi_k^{(b)}$, and discarding all but one rows for each block. Explicitly,

$$A^\pi_{ami,bkj} = \delta_{ab}\delta_{mk}\delta_{ij}(P_m^{(a)} + N_m^{(a)}) + \{(\alpha_a|\alpha_b)\min(t_b m, t_a k) - \delta_{ab}\delta_{mk}\}|\pi_{k,j}^{(b)}| \tag{3.8}$$

where $1 \le i \le l(\pi_m^{(a)})$ and $1 \le j \le l(\pi_k^{(b)})$. From Lemma 3.4 we have $|\mathrm{Sol}'_\pi| = |\det A^\pi|$ if $\det A^\pi \ne 0$.

It is easy to show the following formula by elementary transformations of A^π.

Lemma 3.7.

$$\det A^\pi = \left(\det_{(a,m),(b,k)\in H'}(F_{am,bk})\right) \prod_{(a,m)\in H'} (P_m^{(a)} + N_m^{(a)})^{l(\pi_m^{(a)})-1}. \tag{3.9}$$

Furthermore

Lemma 3.8. *If $N\,(\ne 0) \in \mathcal{N}$ satisfies the condition (2.41), then* $\det A^\pi > 0$.

Proof. By Lemma 3.7, it suffices to verify $\det_{H'} F > 0$. Actually, a stronger statement holds: Let us forget the relation (2.30), and regard $P_m^{(a)}$ in (3.3) as a *nonnegative* integer which is independent of $N_m^{(a)}$'s. Then, $\det_{H'} F > 0$ still holds. We prove the last statement by the double induction on $|H'|$ and the sum $\sum_{(a,m)\in H'} P_m^{(a)}$. First, let H' $(\ne \emptyset)$ be arbitrary, and suppose $\sum_{(a,m)\in H'} P_m^{(a)} = 0$

(i.e., $P^{(a)}_m = 0$ for any $(a,m) \in H'$). Then $\det_{H'} F > 0$ is equivalent to the positivity of $\det_{H'}(\alpha_a|\alpha_b)\min(m/t_a, k/t_b)$, which is a principal minor of the tensor product of two positive-definite matrices, $((\alpha_a|\alpha_b))_{1\le a,b\le n}$ and $(\min(m,k)/s)_{1\le m,k\le L}$ with L and s some integers. Therefore, the claim is true. Next, suppose that $H' = \{(a,m)\}$ (i.e., $|H'| = 1$) and $P^{(a)}_m$ is any nonnegative integer. Then $\det_{H'} F = P^{(a)}_m + (\alpha_a|\alpha_a) m t_a N^{(a)}_m > 0$. Finally, let H' ($|H'| \ge 2$) and $\sum_{(a,m)\in H'} P^{(a)}_m$ (>0) be arbitrary. Then there exists some $(r,i) \in H'$ such that $P^{(r)}_i > 0$. Set $\tilde{P}^{(a)}_m = P^{(a)}_m - 1$ if $(a,m) = (r,i)$, $\tilde{P}^{(a)}_m = P^{(a)}_m$ otherwise, and $\tilde{H}' = H' \setminus \{(r,i)\}$. Then, one can split the determinant as $\det_{H'} F(\{P^{(a)}_m\}) = \det_{\tilde{H}'} F(\{P^{(a)}_m\}) + \det_{H'} F(\{\tilde{P}^{(a)}_m\})$. By the induction hypothesis, the RHS is positive. $\square$

Assembling Lemmas 3.4–3.8, we have

$$\begin{aligned}\frac{|\mathrm{Sol}_{\pi_{\max}}|}{\prod_{(a,m)\in H'} N^{(a)}_m!} &= \frac{1}{\prod_{(a,m)\in H'} N^{(a)}_m!} \sum_{\pi\in\mathcal{L}_N} \mu(\pi,\pi_{\max}) \det A^{\pi} \\ &= \Big(\det_{(a,m),(b,k)\in H'} (F_{am,bk})\Big) \prod_{(a,m)\in H'} \frac{(P^{(a)}_m + N^{(a)}_m)_{N^{(a)}_m}}{N^{(a)}_m!(P^{(a)}_m + N^{(a)}_m)} \\ &= R(\nu, N).\end{aligned}$$

This completes the proof of Theorem 3.2.

4. Generating series

The number $R(\nu,N)$, which was introduced in Section 3 to count the XXZ-type Bethe vectors, is our main concern below. Let us recall the definition of $R(\nu,N)$ in (3.2), where ν, $N \in \mathcal{N}$, and H and $\mathcal{N}$ are defined in (1.2) and (1.14): For $N \ne 0$,

$$R(\nu,N) = \Big(\det_{(a,m),(b,k)\in H'} F_{am,bk}\Big) \prod_{(a,m)\in H'} \frac{1}{N^{(a)}_m} \binom{P^{(a)}_m + N^{(a)}_m - 1}{N^{(a)}_m - 1}, \tag{4.1}$$

$$H' = H'(N) = \{\ (a,m) \in H \mid N^{(a)}_m > 0\ \}, \tag{4.2}$$

$$\gamma^{(a)}_m = \gamma^{(a)}_m(\nu) = \sum_{k=1}^{\infty} \min(m,k)\nu^{(a)}_k, \tag{4.3}$$

$$P^{(a)}_m = P^{(a)}_m(\nu,N) = \gamma^{(a)}_m - \sum_{(b,k)\in H} (\alpha_a|\alpha_b)\min(t_b m, t_a k) N^{(b)}_k, \tag{4.4}$$

$$F_{am,bk} = \delta_{ab}\delta_{mk} P^{(a)}_m + (\alpha_a|\alpha_b)\min(t_b m, t_a k) N^{(b)}_k. \tag{4.5}$$

For $N = 0$, we set $R(\nu,0) = 1$. From now on, we forget the conditions (2.33) and (2.41) required for Theorems 2.11 and 3.2. For any $N \in \mathcal{N}$, $R(\nu,N)$ is an integer (cf. Lemma 4.2), though it is not always a positive one.

On the other hand, the number

$$K(\nu,N) = \prod_{(a,m)\in H} \binom{P^{(a)}_m + N^{(a)}_m}{N^{(a)}_m} \tag{4.6}$$

was introduced in [KR] to count the XXX-type Bethe vectors. Below we treat two numbers, $R(\nu,N)$ and $K(\nu,N)$, in a parallel way so that the relation between them becomes transparent.

4.1. **Generating series.** It is natural to introduce the *generating series* of $R(\nu, N)$ and $K(\nu, N)$,

$$\mathcal{R}^\nu(w) = \sum_{N\in\mathcal{N}} R(\nu, N) w^N, \qquad w^N = \prod_{(a,m)\in H} (w_m^{(a)})^{N_m^{(a)}}, \tag{4.7}$$

$$\mathcal{K}^\nu(w) = \sum_{N\in\mathcal{N}} K(\nu, N) w^N. \tag{4.8}$$

It turns out that the following truncation (projection) is appropriate for our purpose (cf. Remark 4.9): Let l be a fixed nonnegative integer l, and let

$$H_l = \{ (a, m) \mid 1 \le a \le n,\ 1 \le m \le t_a l \}, \tag{4.9}$$

$$\mathcal{N}_l = \{ N = (N_m^{(a)})_{(a,m)\in H} \mid N_m^{(a)} \in \mathbb{Z}_{\ge 0},\ N_m^{(a)} = 0 \text{ for } (a,m) \notin H_l \}, \tag{4.10}$$

so that $\varinjlim H_l = H$ and $\varinjlim \mathcal{N}_l = \mathcal{N}$. The truncated generating series are defined as

$$\mathcal{R}_l^\nu(w) = \sum_{N\in\mathcal{N}_l} R(\nu, N) w^N, \qquad w^N = \prod_{(a,m)\in H_l} (w_m^{(a)})^{N_m^{(a)}}, \tag{4.11}$$

$$\mathcal{K}_l^\nu(w) = \sum_{N\in\mathcal{N}_l} K(\nu, N) w^N. \tag{4.12}$$

Proposition 4.1.

$$\mathcal{R}_l^0(w) = 1.$$

Proof. By definition, $R(0, 0) = 1$. Let $\nu = 0$ and $N \neq 0$. From (4.5), we have $\sum_{(b,k)\in H'} F_{am,bk} = \gamma_m^{(a)}$, and $\gamma_m^{(a)} = 0$ when $\nu = 0$. Thus, $\det F = 0$. Therefore, $R(0, N) = 0$ by (4.1). □

In contrast, $\mathcal{K}_l^0(w)$ is not so simple. See (4.28) and (4.29).

We need the following alternative expression of $R(\nu, N)$.

Lemma 4.2. *For any ν, $N \in \mathcal{N}$, the following equality holds:*

$$R(\nu, N) = \sum_{J\subset H} \left\{ D_J \prod_{(a,m)\in H} \binom{P[J]_m^{(a)} + N[J]_m^{(a)}}{N[J]_m^{(a)}} \right\}, \tag{4.13}$$

where the sum is taken over all the finite subsets J of H,

$$D_J = \begin{cases} 1 & \text{if } J = \emptyset \\ \det_{(a,m),(b,k)\in J} D_{am,bk} & \text{otherwise,} \end{cases} \tag{4.14}$$

$$D_{am,bk} = (\alpha_a|\alpha_b) \min(t_b m, t_a k) - \delta_{ab}\delta_{mk}, \tag{4.15}$$

$P[J]_m^{(a)} := P_m^{(a)}(\nu[J], N[J])$ *with*

$$\nu[J] = (\nu[J]_m^{(a)}), \qquad \nu[J]_m^{(a)} = \nu_m^{(a)} - \sum_{(b,k)\in J} (\alpha_a|\alpha_b) B_{bk,am}, \tag{4.16}$$

$$N[J] = (N[J]_m^{(a)}), \qquad N[J]_m^{(a)} = N_m^{(a)} - \theta((a, m) \in J), \tag{4.17}$$

$B_{bk,am}$ *is defined in (1.3), and* $\theta(\text{true}) = 1$, $\theta(\text{false}) = 0$.

Proof. If $N = 0$, then the both hand sides of (4.13) is 1. Suppose that $N \neq 0$. By (4.5) and (4.15), we have $F_{am,bk} = \delta_{ab}\delta_{bk}(P_m^{(a)} + N_m^{(a)}) + D_{am,bk}N_k^{(b)}$. By splitting the sum $(P_m^{(a)} + N_m^{(a)}) + D_{am,am}N_m^{(a)}$ of each diagonal element in $\det F$ in (4.1), $R(\nu, N)$ is written as

$$\sum_{J\subset H'} \left\{ D_J \prod_{(a,m)\in H'\setminus J} \binom{P_m^{(a)} + N_m^{(a)}}{N_m^{(a)}} \prod_{(a,m)\in J} \binom{P_m^{(a)} + N_m^{(a)} - 1}{N_m^{(a)} - 1} \right\}. \tag{4.18}$$

Then, (4.13) follows from (4.18) and the fact

$$\begin{aligned} P[J]_m^{(a)} - P_m^{(a)} = &-\sum_{j=1}^{\infty} \sum_{(b,k)\in J} \min(m,j)(\alpha_a|\alpha_b)B_{bk,aj} \\ &+ \sum_{(b,k)\in H} (\alpha_a|\alpha_b)\min(t_b m, t_a k)\theta((b,k)\in J) = 0, \end{aligned} \tag{4.19}$$

where the last equality in (4.19) is due to (A.8). □

Remark 4.3. Let

$$\begin{aligned} \tilde{\mathcal{N}} = \{\, N = (N_m^{(a)})_{(a,m)\in H} \mid N_m^{(a)} \in \mathbb{Z},\ N_m^{(a)} = 0 \text{ except for} \\ \text{finitely-many } (a,m) \,\}. \end{aligned} \tag{4.20}$$

We extend the definition of $R(\nu, N)$ to $N \in \tilde{\mathcal{N}}$ such that $R(\nu, N) = 0$ if $N \in \tilde{\mathcal{N}} \setminus \mathcal{N}$. Then, the equality (4.13) still holds for any $N \in \tilde{\mathcal{N}}$ because the both hand sides of (4.13) is 0 if $N \in \tilde{\mathcal{N}} \setminus \mathcal{N}$. We use this fact in the proof of Lemma 5.12.

For $N \in \mathcal{N}_l$, the expression (4.13) reads as

Lemma 4.4. *For $N \in \mathcal{N}_l$, we have*

$$R(\nu, N) = \sum_{J\subset H_l} \left\{ D_J \prod_{(a,m)\in H_l} \binom{P[J]_m^{(a)} + N[J]_m^{(a)}}{N[J]_m^{(a)}} \right\}. \tag{4.21}$$

Proposition 4.5. *$\mathcal{R}_l^\nu(w)$ and $\mathcal{K}_l^\nu(w)$ converge for $|w_m^{(a)}| < (2m-1)^{2m-1}/(2m)^{2m}$.*

Proof. First, we consider $\mathcal{K}_l^\nu(w)$. For given $(a, m) \in H_l$ and $N \in \mathcal{N}_l$, let $N' \in \mathcal{N}_l$ be $N_k'^{(b)} = N_k^{(b)} + \delta_{ab}\delta_{mk}$, and $P_k'^{(b)} = P_k^{(b)}(\nu, N')$. Then, it is easy to check that

$$\lim_{N_m^{(a)}\to\infty} \binom{P_k^{(b)} + N_k^{(b)}}{N_k^{(b)}} \bigg/ \binom{P_k'^{(b)} + N_k'^{(b)}}{N_k'^{(b)}} = \begin{cases} -\frac{(2m-1)^{2m-1}}{(2m)^{2m}} & (b,k) = (a,m) \\ 1 & (b,k) \neq (a,m). \end{cases}$$

Therefore, $\mathcal{K}_l^\nu(w)$ converges for $|w_m^{(a)}| < (2m-1)^{2m-1}/(2m)^{2m}$. Next, consider $\mathcal{R}_l^\nu(w)$. By Lemma 4.4, $\mathcal{R}_l^\nu(w)$ is a linear sum of the power series whose coefficient of w^N is $\prod_{(a,m)\in H_l} \binom{P[J]_m^{(a)}+N[J]_m^{(a)}}{N[J]_m^{(a)}}$. Again, each series converges for $|w_m^{(a)}| < (2m-1)^{2m-1}/(2m)^{2m}$. □

4.2. **Basic identity.** To proceed, we use an identity found by [K2] for A_n and generalized to X_n by [HKOTY].

Let $v = (v_m^{(a)})_{(a,m)\in H_l}$ and $z = (z_m^{(a)})_{(a,m)\in H_l}$ be complex multivariables. We define a map $z = z(v)$ as

$$z_m^{(a)}(v) = v_m^{(a)} \prod_{\substack{(b,k)\in H_l \\ t_b m > t_a k}} (1 - v_k^{(b)})^{(\alpha_a|\alpha_b)(t_b m - t_a k)}. \tag{4.22}$$

The Jacobian $\partial z/\partial v$ is 1 at $v = 0$, so that the map $z(v)$ is biholomorphic around $v = z = 0$. Let $v = v(z)$ the inverse map around $z = 0$.

Lemma 4.6 ([K2, HKOTY])**.** *Let $\beta_m^{(a)}$ $((a,m) \in H_l)$ be arbitrary complex numbers. We have the following power series expansion at $z = 0$ which converges for $|z_m^{(a)}| < 1$.*

$$\begin{aligned}
&\prod_{(a,m)\in H_l} \left(1 - v_m^{(a)}(z)\right)^{-\beta_m^{(a)}-1} \\
&\qquad = \sum_{N\in\mathcal{N}_l} \prod_{(a,m)\in H_l} \binom{\beta_m^{(a)} + c_m^{(a)} + N_m^{(a)}}{N_m^{(a)}} \left(z_m^{(a)}\right)^{N_m^{(a)}},
\end{aligned} \tag{4.23}$$

$$c_m^{(a)} = \sum_{\substack{(b,k)\in H_l \\ t_b m < t_a k}} (\alpha_a|\alpha_b)(t_a k - t_b m) N_k^{(b)}. \tag{4.24}$$

A proof of Lemma 4.6 is given in Appendix B for reader's convenience.

4.3. **Analytic formula.** Let $w = (w_m^{(a)})_{(a,m)\in H_l}$ be another complex multivariable. We define a biholomorphic map $w = w(v)$ around $v = w = 0$ as

$$w_m^{(a)}(v) = v_m^{(a)} \prod_{(b,k)\in H_l} (1 - v_k^{(b)})^{-(\alpha_a|\alpha_b)\min(t_b m, t_a k)}. \tag{4.25}$$

Combining it with (4.22), we have biholomorphic maps among variables v, z, and w around $v = z = w = 0$. Each map is denoted by $v = v(w)$, $z = z(w)$, *etc.* By (4.22) and (4.25), we have the relation

$$z_m^{(a)}(w) = w_m^{(a)} \prod_{(b,k)\in H_l} (1 - v_k^{(b)}(w))^{(\alpha_a|\alpha_b)t_b m}. \tag{4.26}$$

Theorem 4.7. *The following equalities (power series expansions) hold around $w = 0$:*

$$\mathcal{K}_l^\nu(w) = \mathcal{K}_l^0(w) \prod_{(a,m)\in H_l} (1 - v_m^{(a)}(w))^{-\gamma_m^{(a)}}, \tag{4.27}$$

$$\mathcal{K}_l^0(w) = \det_{H_l}\left(\frac{w_k^{(b)}}{v_m^{(a)}} \frac{\partial v_m^{(a)}}{\partial w_k^{(b)}}(w)\right) \prod_{(a,m)\in H_l} (1 - v_m^{(a)}(w))^{-1} \tag{4.28}$$

$$= \left(\det_{H_l}\left(\delta_{ab}\delta_{mk} + D_{am,bk} v_m^{(a)}(w)\right)\right)^{-1}, \tag{4.29}$$

$$\mathcal{R}_l^\nu(w) = \prod_{(a,m)\in H_l} (1 - v_m^{(a)}(w))^{-\gamma_m^{(a)}}, \tag{4.30}$$

where $\gamma^{(a)}_m$ and $D_{am,bk}$ are defined in (4.3) and (4.15).

Proof. (4.27) and (4.28). Let dz abbreviate $\bigwedge_{(a,m)\in H_l} dz^{(a)}_m$. By Lemma 4.6, we have

$$
\begin{aligned}
&\prod_{(a,m)\in H_l} \binom{\beta^{(a)}_m + c^{(a)}_m + N^{(a)}_m}{N^{(a)}_m} \\
&= \operatorname*{Res}_{z=0} \left(\prod_{(a,m)\in H_l} (1 - v^{(a)}_m(z))^{-\beta^{(a)}_m - 1} (z^{(a)}_m)^{-N^{(a)}_m - 1} \right) dz
\end{aligned}
\tag{4.31}
$$

In (4.31), we set $\beta^{(a)}_m = P^{(a)}_m(\nu, N) - c^{(a)}_m = \gamma^{(a)}_m - \sum_{(b,k)\in H_l} (\alpha_a|\alpha_b) t_a k N^{(b)}_k$. After the substitution of (4.26) and the change of the integration variable, we obtain

$$
\begin{aligned}
&\prod_{(a,m)\in H_l} \binom{P^{(a)}_m + N^{(a)}_m}{N^{(a)}_m} \\
&= \operatorname*{Res}_{w=0} \Bigg\{ \left(\prod_{(a,m)\in H_l} (1 - v^{(a)}_m(w))^{-\gamma^{(a)}_m - 1} (w^{(a)}_m)^{-N^{(a)}_m - 1} \right) \\
&\qquad \times \det_{H_l} \left(\frac{w^{(b)}_k}{z^{(a)}_m} \frac{\partial z^{(a)}_m}{\partial w^{(b)}_k}(w) \right) \Bigg\} dw.
\end{aligned}
\tag{4.32}
$$

Also, by (4.22), we have

$$
\det_{H_l} \left(\frac{z^{(b)}_k}{v^{(a)}_m} \frac{\partial v^{(a)}_m}{\partial z^{(b)}_k} \right) = 1,
\tag{4.33}
$$

The equalities (4.27) and (4.28) follows from (4.32), (4.33), and the fact $\gamma = 0$ if $\nu = 0$.

(4.29). By using (4.25), the RHS of (4.28) is easily calculated as (4.29).

(4.30). In (4.32), replace $N^{(a)}_m$ and $P^{(a)}_m$ in the both hand sides by $N[J]^{(a)}_m$ and $P[J]^{(a)}_m$. Accordingly, $\gamma^{(a)}_m$ in the RHS in (4.32) should be also replaced by $\gamma^{(a)}_m - \sum_{(b,k)\in J} (\alpha_a|\alpha_b) \min(t_b m, t_a k)$ (cf. (4.19) and (A.8)). Then, using (4.25), we obtain

$$
\begin{aligned}
&\prod_{(a,m)\in H_l} \binom{P[J]^{(a)}_m + N[J]^{(a)}_m}{N[J]^{(a)}_m} \\
&= \operatorname*{Res}_{w=0} \Bigg\{ \left(\prod_{(a,m)\in H_l} (1 - v^{(a)}_m(w))^{-\gamma^{(a)}_m} (w^{(a)}_m)^{-N^{(a)}_m - 1} \right) \\
&\qquad \times \left(\prod_{(a,m)\in J} v^{(a)}_m(w) \right) K^0_l(w) \Bigg\} dw.
\end{aligned}
\tag{4.34}
$$

The equality (4.30) follows from Lemma 4.4, (4.29), (4.34), and the identity

$$
\sum_{J\subset H_l} \left(D_J \prod_{(a,m)\in J} v^{(a)}_m(w) \right) = \det_{(a,m),(b,k)\in H_l} \left(\delta_{ab}\delta_{mk} + D_{am,bk} v^{(a)}_m(w) \right).
$$

□

Corollary 4.8.

$$\mathcal{R}_l^{\nu}(w) = \mathcal{K}_l^{\nu}(w)/\mathcal{K}_l^0(w), \tag{4.35}$$

$$\mathcal{R}_l^{\nu}(w)\mathcal{R}_l^{\nu'}(w) = \mathcal{R}_l^{\nu+\nu'}(w). \tag{4.36}$$

Proof. (4.35) is immediately obtained from (4.30) and (4.27). (4.36) follows from (4.30) and the fact $\gamma(\nu+\nu')_m^{(a)} = \gamma(\nu)_m^{(a)} + \gamma(\nu')_m^{(a)}$. □

Remark 4.9. Let $\mathbb{C}_l[[w]]$ be the (formal) power series ring of $(w_m^{(a)})_{(a,m)\in H_l}$. There are natural surjections $\varphi_{lk} : \mathbb{C}_k[[w]] \to \mathbb{C}_l[[w]]$ $(l \le k)$ with $\varphi_{lk}(w_m^{(a)}) = 0$ for $(a,m) \in H_k \setminus H_l$. Since $\varphi_{lk}(\mathcal{R}_k^{\nu}(w)) = \mathcal{R}_l^{\nu}(w)$, $(\mathcal{R}_l^{\nu}(w))_{l=1}^{\infty}$ defines an element in the projective limit $\mathbb{C}[[w]] := \varprojlim \mathbb{C}_l[[w]]$, which is identified with $\mathcal{R}^{\nu}(w)$ in (4.7) (and so is $\mathcal{K}^{\nu}(w)$). Then, the equalities (4.30), (4.27), and (4.36) can be rephrased as follows:

$$\mathcal{R}^{\nu}(w) = \mathcal{K}^{\nu}(w)/\mathcal{K}^0(w) = \prod_{(a,m)\in H} (1 - v_m^{(a)}(w))^{-\gamma_m^{(a)}}, \tag{4.37}$$

$$\mathcal{R}^{\nu}(w)\mathcal{R}^{\nu'}(w) = \mathcal{R}^{\nu+\nu'}(w), \tag{4.38}$$

where the RHS of (4.37) denotes $(f_l(w)) \in \mathbb{C}[[w]]$ with $f_l(w)$ being the power series expansion of the the RHS of (4.30) around $w = 0$.

5. Formal completeness of Bethe vectors

Now we are ready to present the main results briefly explained in Section 1.5. Our goal here is to express the coefficient r_λ^{ν} in (1.20) as a sum of the numbers $R(\nu, N)$.

5.1. **Specialization of generating series.** We remind that $y_a = e^{-\alpha_a}$ and $x_a = e^{\Lambda_a}$ are the formal exponents of simple roots and fundamental weights of X_n. We also regard $y = (y_a)$ and $x = (x_a)$ as complex multivariables related by the map $y = y(x)$, $y_a = \prod_{b=1}^{n} x_b^{-(\alpha_a|\alpha_b)t_b}$. We do the specialization $w_m^{(a)}(y) = y_a^m$ of the variable of the series $\mathcal{R}_l^{\nu}(w)$,

$$\mathcal{R}_l^{\nu}(w(y)) = \sum_{N\in\mathcal{N}_l} R(\nu, N) \prod_{a=1}^{n} y_a^{\sum_{m=0}^{\infty} mN_m^{(a)}}. \tag{5.1}$$

The limit

$$\tilde{R}^{\nu}(y) := \lim_{l\to\infty} \mathcal{R}_l^{\nu}(w(y)) = \sum_{N\in\mathcal{N}} R(\nu, N) \prod_{a=1}^{n} y_a^{\sum_{m=0}^{\infty} mN_m^{(a)}} \tag{5.2}$$

exists in $\mathbb{C}[[y]]$, because $\mathcal{R}_l^{\nu}(w(y)) \equiv \tilde{R}^{\nu}(y) \bmod I_l$, where I_l is the ideal of $\mathbb{C}[[y]]$ generated by $y_1^{t_1 l+1}, \dots, y_n^{t_n l+1}$. In the same way, we define the limit

$$\tilde{K}^{\nu}(y) := \lim_{l\to\infty} \mathcal{K}_l^{\nu}(w(y)) = \sum_{N\in\mathcal{N}} K(\nu, N) \prod_{a=1}^{n} y_a^{\sum_{m=0}^{\infty} mN_m^{(a)}}. \tag{5.3}$$

For each $(a,m) \in H$, let $\delta_m^{(a)} = (\nu_k^{(b)})_{(b,k)\in H}$, $\nu_k^{(b)} = \delta_{ab}\delta_{mk}$ and

$$\tilde{R}_m^{(a)}(y) := \tilde{R}^{\delta_m^{(a)}}(y), \quad \tilde{K}_m^{(a)}(y) := \tilde{K}^{\delta_m^{(a)}}(y). \tag{5.4}$$

It immediately follows from Corollary 4.8 that

Proposition 5.1.

$$\tilde{R}^\nu(y) = \tilde{K}^\nu(y)/\tilde{K}^0(y), \quad \tilde{R}^{(a)}_m(y) = \tilde{K}^{(a)}_m(y)/\tilde{K}^0(y), \tag{5.5}$$

$$\tilde{R}^\nu(y)\tilde{R}^{\nu'}(y) = \tilde{R}^{\nu+\nu'}(y), \tag{5.6}$$

$$\tilde{R}^\nu(y) = \prod_{(a,m)\in H} (\tilde{R}^{(a)}_m(y))^{\nu^{(a)}_m}, \tag{5.7}$$

$$\tilde{K}^\nu(y)/\tilde{K}^0(y) = \prod_{(a,m)\in H} (\tilde{K}^{(a)}_m(y)/\tilde{K}^0(y))^{\nu^{(a)}_m}. \tag{5.8}$$

We also introduce the corresponding Laurent series of x as follows:

$$R^{(a)}_m(x) = x_a^m \tilde{R}^{(a)}_m(y(x)), \quad R^\nu(x) = \left(\prod_{(a,m)\in H} x_a^{m\nu^{(a)}_m}\right)\tilde{R}^\nu(y(x)), \tag{5.9}$$

$$K^{(a)}_m(x) = x_a^m \tilde{K}^{(a)}_m(y(x)), \quad K^\nu(x) = \left(\prod_{(a,m)\in H} x_a^{m\nu^{(a)}_m}\right)\tilde{K}^\nu(y(x)). \tag{5.10}$$

Proposition 5.2. *The equalities in (5.5)–(5.8) with $\tilde{R}^\nu$, $\tilde{R}^{(a)}_m$, $\tilde{K}^\nu$, $\tilde{K}^{(a)}_m$ being replaced with R^ν, $R^{(a)}_m$, K^ν, $K^{(a)}_m$ also hold.*

5.2. **Main Results.**

Theorem 5.3. *There exists a unique family $(\tilde{Q}^{(a)}_m)_{(m,a)\in H}$ of invertible power series of y which satisfies ($\tilde{Q}$-I) and ($\tilde{Q}$-II) in Definition 1.3. In fact,*

$$\tilde{Q}^{(a)}_m(y) = \tilde{R}^{(a)}_m(y) = \tilde{K}^{(a)}_m(y)/\tilde{K}^0(y). \tag{5.11}$$

Remark 5.4. The existence of $(\tilde{Q}^{(a)}_m)_{(m,a)\in H}$ also follows from Theorem 1.6 for X_n of classical type. A weak version of the uniqueness was shown in [HKOTY, Theorem 8.1], where the $\mathcal{W}$ invariance of $Q^{(a)}_m$ was further assumed.

A proof of Theorem 5.3 is given in Section 5.3. We present its consequences first. Let $Q^\nu(x)$ be the Laurent series of x defined in (1.16). It follows from Propositions 5.1, 5.2 and (1.16) that

Corollary 5.5.

$$\tilde{Q}^\nu(y) = \tilde{R}^\nu(y) = \tilde{K}^\nu(y)/\tilde{K}^0(y), \tag{5.12}$$

$$Q^\nu(x) = R^\nu(x) = K^\nu(x)/K^0(x). \tag{5.13}$$

Expanding the both hand sides of the first equality in (5.13), we have

$$r^\nu_\lambda = \sum_{N\in\mathcal{N}^\nu_\lambda} R(\nu, N), \tag{5.14}$$

where r^ν_λ is defined in (1.19), and

$$\mathcal{N}^\nu_\lambda = \{\, N\in\mathcal{N} \mid \sum_{(a,m)\in H} m\nu^{(a)}_m\Lambda_a - \sum_{(a,m)\in H} mN^{(a)}_m\alpha_a = \lambda \,\}. \tag{5.15}$$

Therefore,

Corollary 5.6 (Formal completeness of XXZ-type Bethe vectors)**.** *If Conjecture 1.10 is correct, then*

$$\operatorname{ch} W^{\nu}(x) = R^{\nu}(x), \tag{5.16}$$

$$\dim W^{\nu}_{\lambda} = \sum_{N \in \mathcal{N}^{\nu}_{\lambda}} R(\nu, N), \tag{5.17}$$

where $\dim W^{\nu}_{\lambda}$ *denotes the weight multiplicity in* W^{ν} *at weight* λ*.*

Remark 5.7. We know that the number $R(\nu, N)$ correctly counts the Bethe vectors only for special string patterns N. One naive explanation of the equality (5.17) is as follows: $R(\nu, N)$ correctly counts the Bethe vectors, therefore, the weight multiplicity when λ are relatively close to the highest weight of W^{ν}. Then, the factorization property (5.7) imposes such a strong constraint that the equality (5.17) has to hold for the entire region of λ.

With Theorem 1.6, we have

Corollary 5.8. *Let* X_n *be of classical type, and let* $\chi^{(a)}_m$ *be the* X_n*-character given in the RHSs of (1.7)–(1.10). Then*

$$\chi^{(a)}_m(x) = R^{(a)}_m(x) = K^{(a)}_m(x)/K^0(x). \tag{5.18}$$

In particular, $R^{\nu}(x) = K^{\nu}(x)/K^0(x)$ *is a* $\mathcal{W}$*-invariant Laurent polynomial, and* $\tilde{R}^{\nu}(y) = \tilde{K}^{\nu}(y)/\tilde{K}^0(y)$ *is a polynomial.*

Remark 5.9. It remains an open problem to show the $\mathcal{W}$-invariance of $Q^{\nu}(x)$ and the polynomial property of $\tilde{Q}^{\nu}(y)$ for X_n of exceptional type without assuming Conjecture 1.10.

The formal completeness of the XXX-type Bethe vectors has been worked out by [K1, K2, KR, HKOTY]. We reformulate their result in our context as follows (See Appendix C for a proof):

Proposition 5.10. *If* $Q^{(1)}_1(x), \dots, Q^{(n)}_1(x)$ *are* $\mathcal{W}$*-invariant, then*

$$K^0(x) = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha}),$$

where Δ_+ *is the set of all the positive roots of* X_n*.*

If Conjecture 1.10 is correct, then $Q^{(1)}_1(x), \dots, Q^{(n)}_1(x)$ are $\mathcal{W}$-invariant. Then, by Proposition 5.10, we have

$$k^{\nu}_{\lambda} = \sum_{N \in \mathcal{N}^{\nu}_{\lambda}} K(\nu, N), \tag{5.19}$$

where k^{ν}_{λ} is defined in (1.17). Therefore,

Corollary 5.11 (Formal completeness of XXX-type Bethe vectors)**.** *If Conjecture 1.10 is correct, then*

$$\operatorname{ch} W^{\nu}(x) = K^{\nu}(x) / \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha}), \tag{5.20}$$

$$[W^{\nu} : V_{\lambda}] = \sum_{N \in \mathcal{N}^{\nu}_{\lambda}} K(\nu, N), \tag{5.21}$$

where λ is a dominant X_n-weight, and $[W^\nu : V_\lambda]$ denotes the multiplicity of the $U_q(X_n)$-irreducible components V_λ with highest weight λ in W^ν.

5.3. **Proof of Theorem 5.3.**

5.3.1. *Existence.* Because $R(\nu, N=0)=1$, $\tilde{R}^{(a)}_m(y)$ is an invertible power series. We show that $(\tilde{R}^{(a)}_m)$ satisfies ($\tilde{\mathrm{Q}}$-I) and ($\tilde{\mathrm{Q}}$-II) with $\tilde{Q}^{(a)}_m$ being replaced with $\tilde{R}^{(a)}_m$.

($\tilde{\mathrm{Q}}$-I). From the factorization property (5.7), it is enough to prove

Lemma 5.12. *The following relation holds:*

$$\tilde{R}^{\lambda}(y) = \tilde{R}^{\mu}(y) + y_a^m \tilde{R}^{\nu}(y), \tag{5.22}$$

where $\lambda = (\lambda^{(b)}_k)$, $\mu = (\mu^{(b)}_k)$, $\nu = (\nu^{(b)}_k)$ with

$$\begin{aligned} \lambda^{(b)}_k &= 2\delta_{ab}\delta_{mk}, \quad \mu^{(b)}_k = \delta_{ab}(\delta_{m+1,k}+\delta_{m-1,k}),\\ \nu^{(b)}_k &= 2\delta_{ab}\delta_{mk} - (\alpha_a|\alpha_b)B_{am,bk}. \end{aligned}$$

Proof. It is enough to show that

$$R(\lambda, N) = R(\mu, N) + R(\nu, N') \tag{5.23}$$

for $N = (N^{(b)}_k) \in \tilde{\mathcal{N}}$ and $N' = (N'^{(b)}_k) \in \tilde{\mathcal{N}}$ which are related as $N'^{(b)}_k = N^{(b)}_k - \delta_{ab}\delta_{mk}$ ($\tilde{\mathcal{N}}$ is defined in (4.20)). By Remark 4.3, for any $N \in \tilde{\mathcal{N}}$ it holds that

$$R(\lambda, N) = \sum_{J\subset H} \left\{ D_J \prod_{(b,k)\in H} \binom{P^{(b)}_k(\lambda[J], N[J]) + N[J]^{(b)}_k}{N[J]^{(b)}_k} \right\}. \tag{5.24}$$

By (A.6) and (A.8), it is easy to show

$$P^{(b)}_k(\lambda[J], N[J]) = P^{(b)}_k(\mu[J], N[J]) + \delta_{ab}\delta_{mk} = P^{(b)}_k(\nu[J], N'[J]). \tag{5.25}$$

With (5.25), we have

$$\begin{aligned} \binom{P^{(a)}_m(\lambda[J], N[J]) + N[J]^{(a)}_m}{N[J]^{(a)}_m} &= \binom{P^{(a)}_m(\mu[J], N[J]) + N[J]^{(a)}_m}{N[J]^{(a)}_m} \\ &\quad + \binom{P^{(a)}_m(\nu[J], N'[J]) + N'[J]^{(a)}_m}{N'[J]^{(a)}_m}, \end{aligned} \tag{5.26}$$

while for $(b,k) \neq (a,m)$,

$$\begin{aligned} \binom{P[J]^{(b)}_k + N[J]^{(b)}_k}{N[J]^{(b)}_k} &= \binom{P^{(b)}_k(\mu[J], N[J]) + N[J]^{(b)}_k}{N[J]^{(b)}_k} \\ &= \binom{P^{(b)}_k(\nu[J], N'[J]) + N'[J]^{(b)}_k}{N'[J]^{(b)}_k}. \end{aligned} \tag{5.27}$$

The equality (5.23) follows from (5.24), (5.26), and (5.27). □

($\tilde{\mathrm{Q}}$-II). We show the limit $\lim_{m\to\infty} \tilde{R}^{(a)}_m(y)$ exists in $\mathbb{C}[[y]]$. Let $\delta^{(a)}_m$ be the one in (5.4). Then, $P^{(b)}_k(\delta^{(a)}_m, N) = P^{(b)}_k(\delta^{(a)}_{m+1}, N) - \delta_{ab}\theta(k \geq m+1)$ holds from (4.4). In the series $\tilde{R}^{(a)}_m(y)$, those $N = (N^{(b)}_k)$ containing $N^{(a)}_k > 0$ with $k \geq m+1$

make contribution to the power y_a^d only for $d > m$ (see (5.2)). It follows that $\tilde{R}_m^{(a)}(y) \equiv \tilde{R}_{m+1}^{(a)}(y) \bmod y_a^{m+1}\mathbb{C}[[y]]$. Then, we have

$$\tilde{R}_m^{(a)}(y) \equiv \tilde{R}_{m+1}^{(a)}(y) \equiv \tilde{R}_{m+2}^{(a)}(y) \equiv \cdots \mod y_a^{m+1}\mathbb{C}[[y]],$$

which means $\lim_{m\to\infty} \tilde{R}_m^{(a)}(y)$ exists.

5.3.2. *Uniqueness.* Let $(\tilde{Q}_m^{(a)}(y))_{(a,m)\in H}$ be a family of invertible power series of y which satisfies ($\tilde{\text{Q}}$-I) and ($\tilde{\text{Q}}$-II) in Definition 1.3. By Proposition A.2 and ($\tilde{\text{Q}}$-II), the constant term of $\tilde{Q}_m^{(a)}(y)$ is 1. We define a family of power series $(v_m^{(a)}(y))_{(a,m)\in H_l}$ with constant term zero by

$$v_m^{(a)}(y) = 1 - \frac{\tilde{Q}_{m-1}^{(a)}(y)\tilde{Q}_{m+1}^{(a)}(y)}{(\tilde{Q}_m^{(a)}(y))^2}, \quad \tilde{Q}_0^{(a)}(y) = 1. \tag{5.28}$$

We further define a family of power series $(w_m^{(a)}(y))_{(a,m)\in H_l}$ with constant term zero by the composition $w_m^{(a)}(v(y))$ of the series $v_m^{(a)}(y)$ and the power series expansion of the holomorphic map $w = w(v)$ in (4.25) at $v = 0$.

Lemma 5.13. *The following equalities of power series of y hold:*

(5.29)

$$\prod_{(a,m)\in H_l} (1 - v_m^{(a)}(y))^{-\gamma_m^{(a)}} = \left(\prod_{(a,m)\in H_l} (\tilde{Q}_m^{(a)}(y))^{\nu_m^{(a)}}\right)\left(\prod_{a=1}^{n} \frac{(\tilde{Q}_{t_a l}^{(a)}(y))^{\gamma_{t_a l+1}^{(a)}}}{(\tilde{Q}_{t_a l+1}^{(a)}(y))^{\gamma_{t_a l}^{(a)}}}\right),$$

$$w_m^{(a)}(y) = y_a^m \prod_{b=1}^{n}\left(\frac{\tilde{Q}_{t_b l}^{(b)}(y)}{\tilde{Q}_{t_b l+1}^{(b)}(y)}\right)^{(\alpha_a|\alpha_b)t_b m}. \tag{5.30}$$

Proof. (5.29). By rearranging the product indices, the LHS in (5.29) becomes

$$\prod_{a=1}^{n}\left(\frac{(\tilde{Q}_{t_a l}^{(a)}(y))^{\gamma_{t_a l+1}^{(a)}}}{(\tilde{Q}_{t_a l+1}^{(a)}(y))^{\gamma_{t_a l}^{(a)}}} \prod_{m=1}^{t_a l} (\tilde{Q}_m^{(a)}(y))^{2\gamma_m^{(a)} - \gamma_{m+1}^{(a)} - \gamma_{m-1}^{(a)}}\right).$$

By (A.4), we have $2\gamma_m^{(a)} - \gamma_{m+1}^{(a)} - \gamma_{m-1}^{(a)} = \nu_m^{(a)}$.

(5.30). Using the same trick as above and the definition of $B_{am,bk}$ in (1.3), we have

$$\begin{aligned}&\prod_{(b,k)\in H_l} (1 - v_k^{(b)}(y))^{-(\alpha_a|\alpha_b)\min(t_b m, t_a k)}\\ &= \prod_{b=1}^{n}\left(\left(\frac{\tilde{Q}_{t_b l}^{(b)}(y)}{\tilde{Q}_{t_b l+1}^{(b)}(y)}\right)^{(\alpha_a|\alpha_b)t_b m} \prod_{k=1}^{t_b l} (\tilde{Q}_k^{(b)}(y))^{(\alpha_a|\alpha_b)B_{am,bk}}\right).\end{aligned}$$

On the other hand, using ($\tilde{\text{Q}}$-I) and (A.7), we have

$$v_m^{(a)}(y) = y_a^m \prod_{(b,k)\in H_l} (\tilde{Q}_k^{(b)}(y))^{-(\alpha_a|\alpha_b)B_{am,bk}}.$$

for $(a, m) \in H_l$. (5.30) is obtained by multiplying the above two equalities. $\square$

Lemma 5.14. *The following equality of power series of y holds:*

$$\begin{aligned}&\left(\prod_{(a,m)\in H_l}(\tilde{Q}^{(a)}_m(y))^{\nu^{(a)}_m}\right)\left(\prod_{a=1}^{n}\frac{(\tilde{Q}^{(a)}_{t_a l}(y))^{\gamma^{(a)}_{t_a l+1}}}{(\tilde{Q}^{(a)}_{t_a l+1}(y))^{\gamma^{(a)}_{t_a l}}}\right)\\&=\sum_{N\in\mathcal{N}_l}R(\nu,N)\prod_{(a,m)\in H_l}\left(y_a^{mN^{(a)}_m}\prod_{b=1}^{n}\left(\frac{(\tilde{Q}^{(b)}_{t_b l}(y))}{(\tilde{Q}^{(b)}_{t_b l+1}(y))}\right)^{(\alpha_a|\alpha_b)t_b m N^{(a)}_m}\right).\end{aligned}\tag{5.31}$$

Proof. Let us regard (4.30) as an equality of power series of w. Then, by substituting the series $w^{(a)}_m(y)$ for the variable $w^{(a)}_m$ in (4.30) and using (5.29) and (5.30), we obtain (5.31). □

Thanks to the convergence property ($\tilde{\mathrm{Q}}$-II), the limit $l\to\infty$ of (5.31) gives the equality $\prod_{(a,m)\in H}(\tilde{Q}^{(a)}_m(y))^{\nu^{(a)}_m} = \tilde{R}^{\nu}(y)$. In particular, by setting $\nu=\delta^{(a)}_m$, we obtain $\tilde{Q}^{(a)}_m(y)=\tilde{R}^{(a)}_m(y)$. This completes the proof of the uniqueness property of $(\tilde{Q}^{(a)}_m)$, thereby finishes the proof of Theorem 5.3.

Appendix A. Some properties of the function $B_{am,bk}$

The function $B_{am,bk}$,

$$B_{am,bk} := 2\min(t_b m, t_a k) - \min(t_b m, t_a(k+1)) - \min(t_b m, t_a(k-1))\tag{A.1}$$

appears in several places such as (1.4), (1.5), (4.16), (5.22), (5.30), *etc*, and plays a key role. Below we list the properties we use. Since it always appears in the combination $(\alpha_a|\alpha_b)B_{am,bk}$, we are interested only in the situations $(t_a,t_b)=(1,1)$, $(2,2)$, $(3,3)$, $(1,2)$, $(1,3)$, $(2,1)$, $(3,1)$ here.

Let us first observe that the infinite-size matrix $B=(B_{am,bk})_{(a,m),(b,k)\in H}$ is expressed as a product $B=B'D$, where $B'=(B_{am,bk})$, $D=(D_{am,bk})$ with

$$B'_{am,bk}=\min(t_b m, t_a k),\tag{A.2}$$

$$D_{am,bk}=\delta_{ab}(2\delta_{mk}-\delta_{m,k-1}-\delta_{m,k+1}).\tag{A.3}$$

Since the relation

$$2\min(m,k)-\min(m,k+1)-\min(m,k-1)=\delta_{mk}\tag{A.4}$$

holds, the inverse matrix D^{-1} of D is given by

$$(D^{-1})_{am,bk}=\delta_{ab}\min(m,k).\tag{A.5}$$

Proposition A.1. *(i) For each (a,m), there are only finitely-many (b,k)'s such that $B_{am,bk}\neq 0$. Explicitly,*

$$B_{am,bk}=\begin{cases}2\delta_{m,2k}+\delta_{m,2k+1}+\delta_{m,2k-1} & (t_a,t_b)=(2,1)\\ 3\delta_{m,3k}+2\delta_{m,3k+1}+2\delta_{m,3k-1} & (t_a,t_b)=(3,1)\\ \qquad +\delta_{m,3k+2}+\delta_{m,3k-2} & \\ t_a\delta_{t_b m,t_a k} & \text{otherwise.}\end{cases}\tag{A.6}$$

(ii) Let H_l be the subset of H defined in (4.9). Then,

$$B_{am,bk}=0 \quad \text{for } (a,m)\in H_l,\ (b,k)\notin H_l.\tag{A.7}$$

(iii) The following relations hold:

$$\sum_{j=1}^{\infty} B_{am,bj} \min(j,k) = \min(t_b m, t_a k), \tag{A.8}$$

$$\sum_{k=1}^{\infty} B_{am,bk} k = t_b m, \tag{A.9}$$

$$\sum_{(b,k)\in H} (\alpha_a|\alpha_b) B_{am,bk} k \Lambda_b = m\alpha_a. \tag{A.10}$$

Proof. (i) This is shown by the case check. (ii) This can be easily checked by (A.6). (iii) (A.8) is equivalent to the matrix relation $(B'D)D^{-1} = B'$. We have only to care that the matrix product in the LHS is well-defined. This is guaranteed by (i). The LHS of (A.9) can be calculated in a similar way as follows:

$$\sum_{k=1}^{L}\left(\sum_{j=1}^{\infty} B'_{am,bj} D_{bj,bk}\right) k = (L+1)B'_{am,bL} - LB'_{am,bL+1} = t_b m,$$

where L is a sufficiently large number. (A.10) immediately follows from (A.9) and the relation $\alpha_a = \sum_{b=1}^{n} (\alpha_a|\alpha_b) t_b \Lambda_b$. ☐

As an application of (A.6), we show that

Proposition A.2. *The relation ($\tilde{Q}$-I) recursively determines all the other power series $\tilde{Q}^{(a)}_m(y)$ ($m \geq 2$) from given invertible power series $\tilde{Q}^{(1)}_1(y), \ldots, \tilde{Q}^{(n)}_1(y)$ as an initial condition; furthermore, so determined power series $\tilde{Q}^{(a)}_m(y)$ is invertible, and its constant term $c^{(a)}_m$ is the m-th power of the constant term of $\tilde{Q}^{(a)}_1(y)$.*

Proof. We introduce another subset $\mathcal{H}_l$ ($l \geq 1$) of the index set H as

$$\mathcal{H}_l = \{\, (a,m) \in H \mid t(m-1) \leq t_a(l-1) \,\}, \quad t = \max_{1\leq a\leq n} t_a. \tag{A.11}$$

Then, $\mathcal{H}_1 = \{\, (a,1) \mid 1 \leq a \leq n \,\} \subset \mathcal{H}_2 \subset \cdots$ and $\varinjlim \mathcal{H}_l = H$. By ($\tilde{Q}$-I),

$$\tilde{Q}^{(a)}_{m+1}(y) = \frac{(\tilde{Q}^{(a)}_m(y))^2}{\tilde{Q}^{(a)}_{m-1}(y)} \left(1 - y_a^m \prod_{(b,k)\in H} (\tilde{Q}^{(b)}_k(y))^{-(\alpha_a|\alpha_b)B_{am,bk}}\right). \tag{A.12}$$

For a given $(a,m+1) \in H$, let l be a unique positive integer such that $(a,m+1) \in \mathcal{H}_{l+1} \setminus \mathcal{H}_l$. Then, with (A.6), it is easy to check that $B_{am,bk} = 0$ for $(b,k) \notin \mathcal{H}_l$. The claim now follows from (A.12) by induction on l. ☐

Appendix B. Proof of Lemma 4.6

The following proof of Lemma 4.6 is essentially quoted from [HKOTY, Proposition 8.3].

Let $t := \max\{\, t_1, \ldots, t_n \,\}$ and

$$H_l[i] \quad = \quad \{\, (a,m) \in H_l \mid tm \geq t_a i \,\}. \tag{B.1}$$

Then $\emptyset = H_l[tl+1] \subset H_l[tl] \subset \cdots \subset H_l[1] = H_l$. For each $1 \leq i \leq tl$, let $z_i = z_i(v_i)$, $v_i = (v^{(a)}_{m,i})_{(a,m)\in H_l[i]}$, $z_i = (z^{(a)}_{m,i})_{(a,m)\in H_l[i]}$ be the biholomorphic map

around $v_i = z_i = 0$ defined by

$$z^{(a)}_{m,i}(v_i) = v^{(a)}_{m,i} \prod_{(b,k)\in H_l[i]\setminus H_l[tm/t_a]} \Bigl(1 - v^{(b)}_{k,i}\Bigr)^{(\alpha_a|\alpha_b)(t_b m - t_a k)}, \tag{B.2}$$

and $v_i = v_i(z_i)$ be its inverse. ($z^{(a)}_{m,i}$ here corresponds to $z^{(a)}_{m,i-1}$ in [HKOTY].) Let $v_{i+1} = v_{i+1}(v_i)$ be the holomorphic map defined by $v^{(a)}_{m,i+1}(v_i) = v^{(a)}_{m,i}$ (for $(a,m) \in H_l[i+1]$), and $z_{i+1} = z_{i+1}(z_i)$ be the composition $z_{i+1}(v_{i+1}(v_i(z_i)))$. Namely,

$$z^{(a)}_{m,i+1}(z_i) = z^{(a)}_{m,i} \prod_{(b,k)\in H_l[i]\setminus H_l[i+1]} \Bigl(1 - v^{(b)}_{k,i}(z_i)\Bigr)^{-(\alpha_a|\alpha_b)(t_b m - t_a k)}. \tag{B.3}$$

The relation of these variables and maps are summarized by the following diagram:

$$\begin{array}{ccc} v_{i+1} & \leftarrow & v_i \\ \updownarrow & & \updownarrow \\ z_{i+1} & \leftarrow & z_i \end{array}. \tag{B.4}$$

The condition $(b,k) \in H_l[i] \setminus H_l[tm/t_a]$ is equivalent to $(b,k) \in H_l[i]$ and $t_b m > t_a k$. Thus, if we set $z_1 = z$ and $v_1 = v$, the map $z_1(v_1)$ coincides with $z(v)$ in (4.22). Lemma 4.6 is a special case $i = 1$ of the following proposition.

Proposition B.1 ([K2, HKOTY]). *For any integer $1 \le i \le tl$, and any complex numbers $\beta^{(a)}_m$ ($(a,m) \in H_l[i]$), we have the following power series expansion at $z = 0$ which converges for $|z^{(a)}_{m,i}| < 1$:*

$$\begin{aligned} &\prod_{(a,m)\in H_l[i]} \Bigl(1 - v^{(a)}_{m,i}(z_i)\Bigr)^{-\beta^{(a)}_m - 1} \\ &\quad = \sum_{N\in\mathcal{N}_l[i]} \prod_{(a,m)\in H_l[i]} \binom{\beta^{(a)}_m + c^{(a)}_m + N^{(a)}_m}{N^{(a)}_m} \Bigl(z^{(a)}_{m,i}\Bigr)^{N^{(a)}_m}, \end{aligned} \tag{B.5}$$

where

$$c^{(a)}_m = \sum_{(b,k)\in H_l[tm/t_a+1]} (\alpha_a|\alpha_b)(t_a k - t_b m) N^{(b)}_k, \tag{B.6}$$

$$\mathcal{N}_l[i] = \{\, N = (N^{(a)}_m)_{(a,m)\in H} \mid N^{(a)}_m \in \mathbb{Z}_{\ge 0},\ N^{(a)}_m = 0 \text{ for } (a,m) \notin H_l[i] \,\}. \tag{B.7}$$

Remark B.2. If $(a,m) \in H_l[i]$, then $H_l[tm/t_a + 1] \subset H_l[i]$. Also, the condition $(b,k) \in H_l[tm/t_a + 1]$ is equivalent to the condition $(b,k) \in H_l$, $t_b m < t_a k$. Therefore, $c^{(a)}_m$ in (B.6) is the same one as in (4.24).

Proof. We prove the proposition by induction on i in the descent order. First, consider the case $i = tl$. Suppose $(a,m) \in H_l[tl] = \{(a, t_a l)\}_{a=1}^n$. Then, $c^{(a)}_m = 0$ due to $H_l[tm/t_a+1] = H_l[tl+1] = \emptyset$, and $v^{(a)}_{m,tl}(z_{tl}) = z^{(a)}_{m,tl}$ due to $H_l[tl]\setminus H_l[tm/t_a] = \emptyset$. Therefore, the claim reduces to the well-known power series expansion

$$(1 - v)^{-\beta-1} = \sum_{N=0}^{\infty} \binom{\beta + N}{N} v^N, \tag{B.8}$$

which converges for $|v| < 1$. Next, let us assume (B.5) holds for $i+1$. Then, for z_i such that $|z^{(a)}_{m,i}| < 1$ and $|z^{(a)}_{m,i+1}(z_i)| < 1$, the LHS of (B.5) is equal to

$$\begin{aligned}\left(\sum_{N\in\mathcal{N}_l[i+1]}\ \prod_{(a,m)\in H_l[i+1]} \binom{\beta^{(a)}_m + c^{(a)}_m + N^{(a)}_m}{N^{(a)}_m}\left(z^{(a)}_{m,i+1}(z_i)\right)^{N^{(a)}_m}\right)\\ \times\left(\prod_{(a,m)\in H_l[i]\setminus H_i[i+1]} (1-v^{(a)}_{m,i}(z_i))^{-\beta^{(a)}_m-1}\right)\end{aligned} \tag{B.9}$$

by the induction hypothesis. Here, we used the fact that, for $(a,m)\in H_l[i+1]$, $v^{(a)}_{m,i}(z_i) = v^{(a)}_{m,i+1}(v_i(z_i)) = v^{(a)}_{m,i+1}(z_{i+1}(z_i))$. Substituting (B.3) for $z^{(a)}_{m,i+1}(z_i)$ in (B.9), we have

$$\begin{aligned}\sum_{N\in\mathcal{N}_l[i+1]}\left(\prod_{(a,m)\in H_l[i+1]} \binom{\beta^{(a)}_m + c^{(a)}_m + N^{(a)}_m}{N^{(a)}_m}\left(z^{(a)}_{m,i}\right)^{N^{(a)}_m}\right)\\ \times\left(\prod_{(a,m)\in H_l[i]\setminus H_i[i+1]} (1-v^{(a)}_{m,i}(z_i))^{-\beta^{(a)}_m-\tilde{c}^{(a)}_m-1}\right),\end{aligned} \tag{B.10}$$

where

$$\tilde{c}^{(a)}_m = \sum_{(b,k)\in H_l[i+1]} (\alpha_a|\alpha_b)(t_a k - t_b m)N^{(b)}_k. \tag{B.11}$$

Suppose $(a,m)\in H_l[i]\setminus H_l[i+1]$. Then, $tm/t_a = i$ holds. It follows that $\tilde{c}^{(a)}_m = c^{(a)}_m$ and $v^{(a)}_{m,i}(z_i) = z^{(a)}_{m,i}$. Thus, applying (B.8) to the second factor of (B.10), we obtain (B.5). It is easy to check that the RHS in (B.5) converges for $|z^{(a)}_{m,i}| < 1$. □

Appendix C. Proof of Proposition 5.10

Following [HKOTY], we prove the proposition in two steps as Propositions C.3 and C.4. The proof of Proposition C.4 is taken from [HKOTY], while the proof of Proposition C.3 here is new.

Step 1. We start from the formula (4.28), which is also written as (cf. (4.33))

$$\mathcal{K}^0_l(w) = \det_{H_l}\left(\frac{w^{(b)}_k}{z^{(a)}_m}\frac{\partial z^{(a)}_m}{\partial w^{(b)}_k}(w)\right)\prod_{(a,m)\in H_l}(1-v^{(a)}_m(w))^{-1}, \tag{C.1}$$

where $w^{(a)}_m$, $v^{(a)}_m$, and $z^{(a)}_m$ are related by (4.22), (4.25), and (4.26). Let $w^{(a)}_m(y) = y^m_a$ be the specialization in Section 5.1. We define new series of y

$$f_a(y) = \prod_{m=1}^{t_a l}(1-v^{(a)}_m(w(y))),\quad a=1,\,\dots,\,n, \tag{C.2}$$

which depend also on l.

Lemma C.1. *In* $\mathbb{C}[[u]]$,

$$\lim_{l\to\infty} f_a(y) = (\tilde{Q}^{(a)}_1(y))^{-1}. \tag{C.3}$$

Proof. By (4.30), $f_a(y) = \mathcal{R}_l^{\nu}(w(y))$, where $\nu = (\nu_k^{(b)})$ with $\nu_k^{(b)} = -1$ if $(b,k) = (a,1)$, $(a, t_a l)$, 1 if $(b,k) = (a, t_a l + 1)$, and 0 otherwise. Then, the claim follows from the fact that $\mathcal{R}_l^{\nu}(w(y)) \equiv \tilde{R}^{\nu}(y)$ mod I_l, where I_l is the ideal in Section 5.1, and that

$$\lim_{l\to\infty} \tilde{R}^{\nu}(y) = \lim_{l\to\infty} (\tilde{Q}_1^{(a)}(y))^{-1} \frac{\tilde{Q}_{t_a l+1}^{(a)}(y)}{\tilde{Q}_{t_a l}^{(a)}(y)} = (\tilde{Q}_1^{(a)}(y))^{-1} \tag{C.4}$$

by the convergence property (Q̃-II). □

We further define series of y

$$U_a(y) = y_a \prod_{b=1}^{n} \tilde{Q}_1^{(b)}(y)^{-(\alpha_a|\alpha_b)t_b}, \quad a = 1, \ldots, n. \tag{C.5}$$

Lemma C.2. *The following equality of series of y holds:*

$$\tilde{K}^0(y) = \det_{1\le a,b\le n} \left(\frac{y_b}{U_a} \frac{\partial U_a}{\partial y_b}(y) \right) \prod_{a=1}^{n} \tilde{Q}_1^{(a)}(y). \tag{C.6}$$

Proof. We define series of y

$$u_a(y) = y_a \prod_{b=1}^{n} (f_b(y))^{(\alpha_a|\alpha_b)t_b}, \quad a = 1, \cdots, n, \tag{C.7}$$

which depend also on l. By Lemma C.1, (C.6) is the limit $l \to \infty$ of the formula

$$\mathcal{K}_l^0(w(y)) = \det_{1\le a,b\le n} \left(\frac{y_b}{u_a} \frac{\partial u_a}{\partial y_b}(y) \right) \prod_{a=1}^{n} (f_a(y))^{-1}. \tag{C.8}$$

To prove (C.8), we use

$$y_a \frac{\partial}{\partial y_a} = \sum_{m=1}^{t_a l} m w_m^{(a)} \frac{\partial}{\partial w_m^{(a)}}, \tag{C.9}$$

$$\det_{H_l}(\delta_{ab}\delta_{mk} + m\alpha_{abk}) = \det_{1\le a,b\le n}(\delta_{ab} + \sum_{k=1}^{t_b l} k\alpha_{abk}), \quad \alpha_{abk}\text{: arbitrary}, \tag{C.10}$$

where (C.10) is easily shown by elementary transformations. Let

$$F_a(y) = \prod_{b=1}^{n} (f_b(y))^{(\alpha_a|\alpha_b)t_b}. \tag{C.11}$$

By (C.1), (C.8) is equivalent to the following equality:

$$\begin{aligned}
\det_{H_l}\Big(\frac{w_k^{(b)}}{z_m^{(a)}} \frac{\partial z_m^{(a)}}{\partial w_k^{(b)}} \Big) &= \det_{H_l}\Big(\delta_{ab}\delta_{mk} + m w_k^{(b)} \frac{\partial}{\partial w_k^{(b)}} \log F_a \Big) && \text{(by (4.26))} \\
&= \det_{1\le a,b\le n}\Big(\delta_{ab} + \sum_{k=1}^{t_b l} k w_k^{(b)} \frac{\partial}{\partial w_k^{(b)}} \log F_a \Big) && \text{(by (C.10))} \\
&= \det_{1\le a,b\le n}\Big(\delta_{ab} + y_b \frac{\partial}{\partial y_b} \log F_a \Big) && \text{(by (C.9))} \\
&= \det_{1\le a,b\le n}\Big(\frac{y_b}{u_a} \frac{\partial u_a}{\partial y_b} \Big) && \text{(by (C.7)).}
\end{aligned}$$

□

Proposition C.3. *The following equality of Laurent series of x holds:*

$$K^0(x) = \det_{1\le a,b\le n}\left(\frac{\partial Q_1^{(a)}}{\partial x_b}(x)\right). \tag{C.12}$$

Proof. We recall that $y_a(x) = \prod_{b=1}^n x_b^{-(\alpha_a|\alpha_b)t_b}$, $K^0(x) = \tilde{K}^0(y(x))$, and $Q_1^{(a)}(x) = x_a \tilde{Q}_1^{(a)}(y(x))$. Then, by (C.5), we have

$$U_a(y(x)) = \prod_{b=1}^n Q_1^{(b)}(x)^{-(\alpha_a|\alpha_b)t_b}. \tag{C.13}$$

Therefore, by Lemma C.2,

(C.14)

$$K^0(x) = \det_{1\le a,b\le n}\left(\frac{x_b}{Q_1^{(a)}}\frac{\partial Q_1^{(a)}}{\partial x_b}(x)\right)\prod_{a=1}^n \tilde{Q}_1^{(a)}(y(x)) = \det_{1\le a,b\le n}\left(\frac{\partial Q_1^{(a)}}{\partial x_b}(x)\right).$$

□

Step 2. So far, we have not used the assumption of Proposition 5.10 that $Q_m^{(a)}$ are $\mathcal{W}$-invariant yet.

Proposition C.4. *If $Q_1^{(1)}(x), \dots, Q_1^{(n)}(x)$ are $\mathcal{W}$-invariant, then*

$$\det_{1\le a,b\le n}\left(\frac{\partial Q_1^{(a)}}{\partial x_b}(x)\right) = \prod_{\alpha\in\Delta_+}(1-e^{-\alpha}). \tag{C.15}$$

Proof. It is well-known that

$$e^\rho \prod_{\alpha\in\Delta_+}(1-e^{-\alpha}) = \sum_{w\in\mathcal{W}} \operatorname{sgn}(w) e^{w(\rho)},$$

and the RHS is characterized by (i) the coefficient of e^ρ is 1; (ii) it is skew $\mathcal{W}$-invariant. Therefore, it is enough to show that $e^\rho(\partial Q_1/\partial x)(x)$ satisfies the same properties. The property (i) follows from the fact $\tilde{Q}_1^{(a)}(y) = 1 + O(y)$. Under the assumption that $Q_1^{(a)}(x)$ are $\mathcal{W}$-invariant, the property (ii) is equivalent to the fact that $e^{-\rho} dx_1 \wedge \cdots \wedge dx_n$ is skew $\mathcal{W}$-invariant, This is easily seen by using the following well-known transformation property under the simple reflection s_a:

$$s_a(e^{-\rho}) = (y_a)^{-1} e^{-\rho}, \qquad s_a(dx_1\wedge\cdots\wedge dx_n) = -y_a(dx_1\wedge\cdots\wedge dx_n).$$

□

Proposition 5.10 follows from Propositions C.3 and C.4.

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